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# Codensity, Compactness and Ultrafilters

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# Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

*(Barry-Patrick Devlin)*

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# Lay Summary

Category theory is all about things called categories; these are special collections of points and arrows. In this thesis I used category theory to study finiteness. There are many different types of “finiteness” in mathematics, for example compactness and pro-finiteness. I have shown that there is an underlying theory connecting some of these notions together. I have constructed a mathematical machine that when fed a certain notion of finiteness spits out a different type of finite space.

# Abstract

Codensity monads are ubiquitous, as are various different notions of compactness and finiteness. Two such examples of “compact” spaces are compact Hausdorff Spaces and Linearly Compact Vector Spaces. Compact Hausdorff Spaces are the algebras of the codensity monad induced by the inclusion of finite sets in the category of sets. Similarly linearly compact vector spaces are the algebras of the codensity monad induced by the inclusion of finite dimensional vector spaces in the category of vector spaces. So in these two examples the notions of finiteness, compactness and codensity are intertwined. In this thesis we generalise these results. To do this we generalise the notion of ultrafilter, and follow the intuition of the compact Hausdorff case. We give definitions of general notions of “finiteness” and “compactness” and show that the algebras for the codensity monad induced by the “finite” objects are exactly the “compact” objects.

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# Chapter 1

## Introduction

Two well known ideas play a central role in this thesis. The first of these is the notion of an ultrafilter. Ultrafilters appear in logic, set theory and topology. Given a set  $S$ , a filter on  $S$ , is usually thought of as a family of “big” subsets of  $S$ . As supersets and intersections of “big” subsets are also “big”, filters are closed under supersets and intersections. Filters are ordered by inclusion, with this order, an ultrafilter is a maximal filter. So we can think of an ultrafilter as a “biggest” family of “big” subsets.

We can also think of an ultrafilter on a set  $S$  in terms of partitions of  $S$ , as explained by Leinster in [13]. A family  $\mathcal{U}$  of subsets of  $S$  is an ultrafilter on  $S$  if, for any partition of  $S$  into disjoint subsets, exactly one of these disjoint non-empty subsets is in  $\mathcal{U}$ .

The simplest ultrafilters on a set are the principal ultrafilters. These are composed of all the subsets containing a specific point. In fact, on finite sets all ultrafilters are principal. Using the axiom of choice we can show that non-principal ultrafilters exist on infinite sets, unfortunately we cannot construct them explicitly. However, given any family of subsets with the finite intersection property we can, by appeal to Zorn’s lemma, assert the existence of an ultrafilter containing them.

We have an endofunctor  $\mathfrak{U}$  on **Set** that sends a set to the set of ultrafilters on that set. This endofunctor is part of a monad, the ultrafilter monad. In Chapter 3 we will cover the background material on ultrafilters in detail.

The other central idea of this thesis is codensity. The dual notion, density, was introduced by Isbell in [9]. Intuitively, a subcategory  $\mathcal{B}$  of a larger category  $\mathcal{A}$  is dense in  $\mathcal{A}$  if every object of  $\mathcal{A}$  can be constructed as a colimit of objects of  $\mathcal{B}$  in a canonical way.

For example consider the one point set,  $\{*\}$ , this is dense in the category of sets. Given a set  $S$  we have

$$S = \bigsqcup_{s \in S} \{s\}.$$

So we have written  $S$  as a colimit of one point sets. Indeed we can think of elements  $s$  of  $S$  as morphisms from  $\{*\}$  to  $S$ , so we have

$$S = \bigsqcup_{\{*\} \xrightarrow{f} S} \{f(*)\}.$$

that is,  $S$  is a colimit of one point sets indexed by morphisms from  $\{*\}$  to  $S$ .

More rigorously, given an object  $A$  of the category  $\mathcal{A}$ , consider the family of morphisms with domain in the subcategory  $\mathcal{B}$  and codomain  $A$ . This family of morphisms is the canonical cocone from  $\mathcal{B}$  to  $A$ . The subcategory  $\mathcal{B}$  is dense if, for all objects  $A$  in  $\mathcal{A}$ , the canonical cocone from  $A$  into  $\mathcal{B}$  is a colimit cocone.

Dually, we have the canonical cone from  $A$  to  $\mathcal{B}$ , this consists of the morphisms with domain  $A$  and codomain in the subcategory  $\mathcal{B}$ . The subcategory  $\mathcal{B}$  is codense if for all  $A$  in  $\mathcal{A}$  the canonical cone from  $A$  into  $\mathcal{B}$  is a limit cone.

Of course, not all subcategories are codense. If the subcategory  $\mathcal{B}$  is not codense in  $\mathcal{A}$ , we can still form the canonical limit mentioned above, in fact these limits give us an endofunctor  $T$  on  $\mathcal{A}$ . Given an object  $A$  of  $\mathcal{A}$  we define  $TA$  as the limit of the canonical cone from  $A$  into  $\mathcal{B}$ . When  $\mathcal{B}$  is codense in  $\mathcal{A}$  we have  $TA \cong A$  for all objects  $A$  of  $\mathcal{A}$ .

The endofunctor  $T$  can equivalently be defined as the right Kan extension of the inclusion functor  $F : \mathcal{B} \rightarrow \mathcal{A}$  along itself. With this approach  $\mathcal{B}$  is codense if the right Kan extension of  $F$  along itself is the identity. We will also see that, if  $\mathcal{A}$  is a concrete category, then the elements of  $TA$  can be thought of as certain natural transformations.

Suppose that the inclusion functor from  $\mathcal{B}$  to  $\mathcal{A}$  has a left adjoint. This adjunction would induce a monad on  $\mathcal{A}$ . In fact  $T$  will be the endofunctor of this adjunction induced monad. However, even in the case that the inclusion functor does not have a left adjoint  $T$  will be the endofunctor of a monad on  $\mathcal{A}$ . This monad is known as the codensity monad and is equal to the adjunction induced monad when the inclusion has a left adjoint. Codensity monads were introduced independently by Kock in [11] and Appelgate and Tierney in [2]. We cover the background material on codensity and codensity monads in chapter four.

So, given some category  $\mathcal{A}$  with a natural choice of subcategory  $\mathcal{B}$ , the first question we might ask is “what is the codensity monad?”. In particular, we can ask, “what is the endofunctor of the codensity monad?”. Once we understand the codensity monad itself, the most natural question to ask is, “what are the algebras for this monad?”.

My thesis is concerned with categories of algebras such as the category of vector spaces, the category of groups or the category of rings. An algebraic theory usually has a few sensible ideas of finiteness. For example, in the case of sets we have the finite sets, in the case of rings we have the finite rings and the Noetherian rings and in the case of modules we have the finitely generated modules.

As explained above, this subcategory of “finite” algebras will give a codensity monad on the larger category of algebras. In this thesis we will try to answer the above questions, that is, what the codensity monad is and what exactly the algebras for this monad are.

We will consider two different examples. The first is the case of the finite sets in the category of sets. In [10] Kennison and Guindenhuis show that the codensity monad induced by the finite sets in **Set** is the ultrafilter monad.

What are the algebras for this monad? Well, let’s briefly consider ultrafilters on a topological space. We can think of ultrafilters as being a generalisation of sequences. With this in mind, we can define the limit of an ultrafilter analogously. Suppose that  $\mathcal{U}$  is an ultrafilter on the topological space  $S$ , the point  $p \in S$  is a limit point of the ultrafilter  $\mathcal{U}$ , if, for every open set  $U \subset S$  with  $p \in U$  we have  $U \in \mathcal{U}$ . By considering complements this definition is equivalent to saying that  $p$  is a limit point of  $\mathcal{U}$  if it is in the intersection of the closed sets of  $\mathcal{U}$ .

Now, a topological space is compact Hausdorff exactly when every ultrafilter has a unique

limit. This is equivalent to saying that the space  $S$  is compact Hausdorff when we have a limit morphism from the set of ultrafilters on  $S$ ,  $\mathfrak{U}(S)$ , to the set  $S$  itself. So, compact Hausdorff topologies on a set  $S$  correspond with certain special morphisms  $\mathfrak{U}(S) \rightarrow S$ . Indeed, as Manes shows in [15], the compact Hausdorff spaces are exactly the algebras of the ultrafilter monad.

Let's now consider the second example, the case of finite dimensional vector spaces in the category of vector spaces. In [13] Leinster shows that the endofunctor of the codensity monad induced by the finite dimensional vector spaces in **Vect** is double dualisation. Leinster also shows that the algebras for this monad are the linearly compact vector spaces introduced by Lefschetz in [12]. These linear compact vector spaces are the topological vector spaces satisfying the following conditions

1. The topology has a basis of open affine sets
2. The topology is Hausdorff
3. Any family of closed affine sets with the finite intersection property has non-empty intersection.

So, we have two different algebraic theories, sets and vector spaces, with corresponding notions of finiteness, finite sets and finite dimensional vector spaces. In both cases the algebras for the codensity monad induced by the subcategory of finite sets/finite dimensional vector spaces are the topological sets/vector spaces that are in some sense “compact”.

This correspondence is noted by Leinster in [13]. Suppose we had some more general notion of a compact topological algebra of which compact Hausdorff space and linearly compact vector spaces are special cases. Suppose also that we have some general notion of finiteness. In [13], Leinster proposes that the algebras for the codensity monad induced by the subcategory of finite algebras will be these “compact” topological algebras.

In this thesis, we will propose general definitions for the notions of “finiteness” and “compactness” proposed above. We will then show that the algebras of the codensity monad induced by this notion of “finiteness” are in fact the “compact” topological algebras.

We will define what we mean by a subcategory of “finite” algebras in Chapter 2. Then, in Chapter 4 we propose a definition of “compact” topological algebra, an ‘algebraically compact algebra’ in our terminology. To make this definition we need the notion of an affine cofinite set. Given an algebra  $A$  we define the affine cofinite subsets of  $A$  as the subsets of the form  $f^{-1}(b)$  for some morphisms  $A \xrightarrow{f} B$  with  $B$  finite. With this in mind, a topological algebra is algebraically compact if it satisfies the following conditions

1. The topology has a basis of open affine cofinite sets
2. The topology is Hausdorff
3. Any family of affine cofinite sets with the finite intersection property has non-empty intersection.

Our definition of algebraically compact algebras is clearly reminiscent of linearly compact vector spaces. Indeed in the **Vect** case the algebraically compact algebras are linearly compact vector spaces. In the **Set** case, any subset can be written as a fibre of a morphism into the

two element set, so every subset is affine cofinite. With this observation the above definition reduces to that of compact Hausdorff space in the **Set** case.

Our theory, for the most part, generalises the theory (and intuition) of the **Set** case. In particular in the second chapter of this thesis we generalise the common notion of ultrafilter. Ordinary ultrafilters are defined on sets, our more general definition works for algebraic theories such as groups, rings or vector spaces. Ordinary ultrafilters are collections of subsets, however our ultrafilters will be collections of affine cofinite subsets.

Our definition of ultrafilter is a direct generalisation of the “partition” definition of ultrafilters on set. Suppose we have a morphism from a set  $S$  into a finite set  $n$ . Then the fibres of this morphism form a partition of  $S$  into disjoint subsets. Conversely, suppose we have a partition of the set  $S$  into  $n$  disjoint subsets. If we number the subsets of this partition from 1 to  $n$  we can construct a morphism from  $S$  to  $n$ . So, a partition of a set  $S$  into  $n$  finitely many disjoint subsets is the same thing as a morphism from  $S$  to the finite set  $n$ .

Now, a family  $\mathcal{U}$  of subsets of  $S$  is an ultrafilter if, for every partition of  $S$  into finitely many disjoint subsets, exactly one of these disjoint subsets is in  $\mathcal{U}$ . By the above, this is equivalent to requiring that for every morphism  $f$  from  $S$  to a finite set  $n$ , there is a unique element  $m \in n$  such that  $f^{-1}(m)$  is in  $\mathcal{U}$ .

So, we say a family  $\mathcal{U}$  of affine cofinite subsets of an algebra  $A$  is an ultrafilter if, for any morphism  $f$  from  $A$  to a finite algebra  $B$ , there is a unique element  $b$  of  $B$  such that  $f^{-1}(b)$  is in  $\mathcal{U}$ . There is one other technical condition in our definition that we omit for now but will cover in detail later.

As well as defining these more general ultrafilters, in Chapter 3, we prove a number of interesting results about ultrafilters. In particular we show that, as with ultrafilters on sets, they are closed under intersections and (affine cofinite) supersets.

Now, a collection of subsets of some set  $S$  that is closed under intersections and supersets is an ultrafilter if and only if it is maximal among such collections of subsets. We will show that this result holds for our more general ultrafilters. A family of affine cofinite sets, that is closed under intersections and (affine cofinite) supersets is an ultrafilter if and only if it is maximal among such families of affine cofinite sets.

There are two main results in this thesis. The first of these is derived in Chapter 6. Generalising from Manes [15], we show that the codensity monad induced by a subcategory of finite algebras is, as in the **Set** case, the ultrafilter monad.

The second main result is that, with the above definitions, the algebras for the monad induced by the finite algebras are exactly the algebraically compact algebras. This result is spread over chapters 7, 8 and 9. Of the two examples mentioned above, linearly compact vector spaces are a special case of this theory.

Specifically, in chapter 7 we show how to construct an algebra for the codensity monad given an algebraically compact algebra. To do this we generalise from the set case by defining a notion of limit for our more general ultrafilters. Then we show that there is a limit morphism that sends an ultrafilter to its limit and that this morphism is an algebra for the codensity monad.

In the ordinary set case we have seen that there are two equivalent definitions of the limit of an ultrafilter on a topological space. One definition involves open sets and the other closed sets. We use the definition involving closed sets to show that ultrafilters on compact spaces always

have at least one limit and we use the definition involving open sets to show that ultrafilters on Hausdorff spaces have at most one limit. By the equivalence of these definitions we have that ultrafilters on compact Hausdorff spaces have exactly one limit.

We can swap between open and closed sets easily in the set case. However this is not true when we are considering affine cofinite sets, as the complement of an affine cofinite set need not be affine cofinite. There is however a way we can avoid this problem. We do so by defining a limit of an ultrafilter as a point in the intersection of the clopen sets of that ultrafilter. Just as in the set case, we use the third condition of algebraic compactness, which is most similar to the regular notion of compactness, to show that every ultrafilter on an algebraically compact algebra has at least one limit and we use the second condition, Hausdorffness, to show that every ultrafilter on an algebraically compact algebra has at most one limit. However we must restrict to theories containing the theory of groups, such as rings or vector spaces. These theories will have “enough” clopen sets for this definition to work.

Working in the opposite direction, in Chapter 8 we show that given an algebra for the codensity monad we can construct an algebraically compact topology. We define our topology to be the topology for which our algebra would be the limit morphism. So given an algebra  $TA \xrightarrow{h} A$  we say that an affine cofinite subset  $S$  of  $A$  is open if for all ultrafilters  $\mathcal{U}$  on  $A$ , whenever  $h(\mathcal{U})$  is in  $S$  we have  $S \in \mathcal{U}$ .

We need to show that this topology is compatible with the algebraic structure of our theory. We first show that, if the group operation is multiplication, then multiplication by a fixed element is continuous, we then use this to show that all the operations of our theory are continuous.

It is quite easy to see that this topology satisfies the third condition of algebraic compactness. We can show that the point  $h(\mathcal{U})$  is in every closed set of an ultrafilter  $\mathcal{U}$ . So, given a family of closed affine cofinite sets with the finite intersection property, we can extend them to an ultrafilter  $\mathcal{U}$ . The point  $h(\mathcal{U})$  will be in every closed set of the ultrafilter  $\mathcal{U}$ , so in particular,  $h(\mathcal{U})$  will be in every one of the original family of closed affine cofinite sets, so their intersection will be non-empty.

Showing that our topology is Hausdorff is significantly more difficult. On an ordinary topological space we can show that a point  $p$  is in the closure of a set  $S$  if and only if there is an ultrafilter  $\mathcal{U}$  with  $S \in \mathcal{U}$  for which  $p$  is a limit point. We show that this is also true for the general case and use this fact to prove that our topology is Kolmogorov. However for theories that contain the theory of groups, this is equivalent to being Hausdorff.

So, we show that given an algebraically compact algebra we can construct a  $T$ -algebra and given a  $T$ -algebra we can construct an algebraically compact algebra. In Chapter 9 we show that these constructions are mutually inverse and prove that we in fact have an isomorphism of categories, between the category of algebras for the codensity monad and the category of algebraically compact algebras. An algebra for the codensity monad induces an algebraically compact topology for which it would be the limit, and an algebraically compact topology has a limit which is an algebra for the codensity monad.

We have mentioned two examples of this theory, compact Hausdorff spaces and linearly compact vector spaces. In Chapter 10 we give two more, otherwise well known, examples, profinite groups and profinite rings. We show that the codensity monad induced by the finite groups in the category of groups is the profinite completion monad for groups. We also show

that the algebraically compact groups are the profinite groups. So, using the main result of this thesis, the algebras for the profinite completion monad for groups are the profinite groups.

In exactly the same manner we see that the codensity monad induced by the finite rings in the category of rings is the profinite completion monad for rings and the algebraically compact rings are the profinite rings. So, using the main result of this thesis once again, the algebras for the profinite completion monad for rings are the profinite rings.

There are a number of open questions. Firstly, can this theory be generalised to a broader class of theories, in particular those that do not contain the theory of groups? Much of our theory relies on the idea of the generalised ultrafilter, it was necessary for us to restrict to theories containing the theory of groups in order for the notion of limit of an ultrafilter to have the properties we would like. If the theory could be developed without use of generalised ultrafilters it could potentially apply to a broader range of algebraic theories.

Although we relied on the intuition of the set case, as the theory of sets does not contain the theory of groups, our theory does not apply to sets. So it is an open question if it is possible to unify the theory of compact Hausdorff spaces with the theory of algebraically compact algebras.

The second obvious question is, can our notion of finiteness be improved? We have chosen certain restrictions for a subcategory of finite algebras so that our definition of ultrafilters behaves analogously to the conventional set theory definition. Once again, if this generalisation of the **Set** case could be avoided it might be possible to give a less restrictive list of requirements for a subcategory of finite algebras.

The word ‘algebra’ is used with two distinct meanings which could potentially cause confusion. We use it both in the sense of a model of an algebraic theory and an algebra for a monad. We also use the word ‘finite’ with two distinct meanings in mind. Either we are referring to the objects of the subcategory  $\mathcal{B}$  of algebras that we consider finite in a generalised sense or we are referring to algebras that have finitely many elements in the conventional sense. Hopefully from the context the intended meaning will be clear. Standard categories such as the category of sets or the category of vector spaces are written in bold, for example **Set** and **Vect**. General categories are denoted by calligraphic letters and their objects by upper case latin letters. Upper case latin letters also denote functors and lower case latin letters denote morphisms.

## Chapter 2

# Definitions

We begin by making a number of basic definitions and proving that the image of an affine cofinite set is also affine cofinite, a result that we will rely on later.

### 2.1 Algebras

In this thesis we will be considering finitary algebraic theories. Informally an algebraic theory is a theory with function symbols but no relations and only equations as axioms. The axioms of an algebraic theory cannot contain inequalities or quantifiers. Groups, lattices and rings are all algebraic theories. An algebraic theory is finitary if the operations of the theory have only finitely many variables. Algebraic Theories and finitary algebraic theories are defined rigorously by Manes in [16] and by Adámek, Rosický and Vitale in [1]. A standard reference for this topic is [3].

When we specify a list of operations and axioms for a theory we are giving a presentation of that theory. For example, we can define the theory of groups as the theory with one unary operation  $-^{-1}$ , one nullary operation  $1$  and one binary operation  $- \cdot -$  and the axioms

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

$$x \cdot x^{-1} = 1 = x^{-1} \cdot x$$

and

$$x \cdot 1 = x = 1 \cdot x.$$

However, a theory can have more than one presentation, for example see the unusual presentation of a group given at the beginning of Chapter 1 in [16].

We will primarily be considering finitary theories that contain the theory of groups.

**Definition 2.1.1.** *An algebraic theory that **contains the theory of groups**, is a theory  $T$  together with a morphism of theories from the theory of groups to  $T$ .*

In other words, an algebraic theory contains the theory of groups if it has a presentation containing the group operations and axioms described above. That is, there is a distinguished forgetful functor from our category of algebras to the category of groups. The theories of vector spaces, rings and abelian groups all contain the theory of groups.

For the rest of this chapter, and generally in this thesis, we will suppose that for some theory containing the theory of groups, we have a category  $\mathcal{A}$  of algebras and algebra homomorphisms. We will denote the identity element of the underlying group structure by 1 and the group operation by  $\cdot$ .

**Definition 2.1.2.** A *topological algebra* is an algebra which also has the structure of a topological space such that all the operations of the algebra are continuous with respect to this topology.

Morphisms of topological algebras are algebra morphisms that are also continuous morphisms. Topological vector spaces and topological groups are common examples of topological algebras.

## 2.2 Affine Sets

In a vector space there are certain special subsets which are called affine subsets, intuitively these are points, lines, planes and their higher dimensional analogues.

**Definition 2.2.1.** The *affine subsets* of a vector space are the cosets of the linear subspaces and the empty subspace.

We generalise this notion to arbitrary algebras.

**Definition 2.2.2.** A subset  $S$  of an algebra  $A$  is *affine* if there exists an algebra  $B$  in  $\mathcal{A}$ , a morphism  $A \xrightarrow{f} B$  in  $\mathcal{A}$  and an element  $b \in B$  with  $S = f^{-1}(b)$ .

Suppose that  $\mathcal{A}$  is the category **Set**. Given a set  $S$ , every subset of  $S$  can be written as a fibre of a morphism into the 2 element set. So, in this case, every subset is an affine subset.

Suppose alternatively that  $\mathcal{A}$  is the category **Vect** - the category vector spaces and linear maps. If we have a morphism  $A \xrightarrow{f} B$  and a  $b$  in  $B$  then  $f^{-1}(b)$  will either be empty or it will be coset of a linear subspace of  $A$ , so this definition reduces to the ordinary sense of affine subspace defined above.

We now prove some simple facts about affine subsets.

**Lemma 2.2.3.** Given algebras  $A$  and  $B$  and a morphism  $A \xrightarrow{f} B$ , if  $S$  is an affine subset of  $B$ , then  $f^{-1}(S)$  will be an affine subset of  $A$ .

*Proof.* As  $S$  is affine we have a morphism  $B \xrightarrow{h} C$  and a  $c \in C$  with  $S = h^{-1}(c)$ . This gives

$$f^{-1}(S) = f^{-1}(h^{-1}(c)) = (hf)^{-1}(c)$$

so  $S$  is affine. □

**Lemma 2.2.4.** Affine subsets of an algebra are closed under finite intersections.

*Proof.* Suppose we have an algebra  $A$ . First we consider the empty intersection, that is, the intersection of no affine subsets of  $A$ , this will just be the set  $A$  itself. Consider the morphism from  $A$  to the terminal algebra,  $A$  is the only fibre of this morphism so  $A$  is an affine subset of  $A$ . That is, the empty intersection is an affine subset of  $A$ .



Suppose  $U$  and  $V$  are affine subsets of  $A$ . So we have morphisms  $A \xrightarrow{g} B$  and  $A \xrightarrow{h} C$  and elements  $b \in B$  and  $c \in C$  such that  $U = g^{-1}(b)$  and  $V = h^{-1}(c)$ . Consider the product morphism

$$\begin{array}{ccccc} & & A & & \\ & f \swarrow & \downarrow \langle f, g \rangle & \searrow g & \\ B & \xleftarrow{p} & B \times C & \xrightarrow{q} & C \end{array}$$

We have,  $x \in V \cap W$  if and only if,  $g(x) = b$  and  $h(x) = c$ , which is equivalent to  $x \in \langle g, h \rangle^{-1}(\langle b, c \rangle)$ . So we have

$$V \cap W = \langle g, h \rangle^{-1}(\langle b, c \rangle)$$

and so  $V \cap W$  is affine. So, by induction, finite intersections of affine sets are affine.  $\square$

As we assume that the algebraic theories that we consider contain the theory of groups we can usefully classify the affine subsets as follows.

**Lemma 2.2.5.** *Suppose that we have algebras  $A$  and  $B$  and a morphism  $A \xrightarrow{h} B$ , then for  $b$  and  $c$  in  $B$  we have*

$$b \cdot h^{-1}(c) = h^{-1}(h(b) \cdot c).$$

*Proof.* First suppose that  $h^{-1}(c)$  is empty. Then  $b \cdot h^{-1}(c)$  is also empty, so we have  $b \cdot h^{-1}(c) \subset h^{-1}(h(b) \cdot c)$ . Now, suppose that we have  $a \in b \cdot h^{-1}(c)$  then we have  $a = b \cdot d$  for some  $d$  with  $h(d) = c$ . Then we have

$$h(a) = h(b \cdot d) = h(b) \cdot h(d) = h(b) \cdot c$$

so  $a \in h^{-1}(h(b) \cdot c)$ , that is  $b \cdot h^{-1}(c) \subset h^{-1}(h(b) \cdot c)$ .

In the other direction, suppose that  $h^{-1}(h(b) \cdot c)$  is empty. Then we have  $h^{-1}(h(b) \cdot c) \subset b \cdot h^{-1}(c)$ . On the other hand, suppose we have  $a' \in h^{-1}(h(b) \cdot c)$ . We have  $a' = b \cdot b^{-1} \cdot a'$  and

$$h(b^{-1} \cdot a') = h(b)^{-1} \cdot h(a') = h(b)^{-1} \cdot h(b) \cdot c = c$$

so  $a' \in b \cdot h^{-1}(c)$ . This gives  $h^{-1}(h(b) \cdot c) \subset b \cdot h^{-1}(c)$ , that is,  $h^{-1}(h(b) \cdot c) = b \cdot h^{-1}(c)$ .  $\square$

In particular, if in the previous Lemma we let  $c$  be  $1_B$ , then we have that

$$b \cdot h^{-1}(1_B) = h^{-1}(h(b)).$$

So the non-empty fibres of the morphism  $h$  are just the cosets of the fibre  $h^{-1}(1_B)$ .

## 2.3 Finite Algebras

In this thesis we will be considering categories of algebras that come with a notion of “finiteness”. In this section we define what we mean by “finiteness”. First we make the following two definitions.

**Definition 2.3.1.** *Given a category  $\mathcal{C}$ , a subcategory  $\mathcal{B}$  of  $\mathcal{C}$  is a **pseudovariety** if  $\mathcal{B}$  is*

1. *Closed under taking subalgebras*

2. Closed under taking quotients

3. Closed under taking finite products.

In [6], Eilenberg and Schützenberger define pseudovarieties to help generalise Birkhoff’s theorem. Below, we rephrase a familiar notion in terms of affine subsets.

**Definition 2.3.2.** *An algebra satisfies the **affine descending chain condition** if every descending chain of affine sets stabilises.*

With the above notions we can now define our notion of “finiteness”.

**Definition 2.3.3.** *Given a category of algebras  $\mathcal{A}$ , a small full subcategory  $\mathcal{B}$  of  $\mathcal{A}$  is a subcategory of **finite** algebras if*

- *The subcategory  $\mathcal{B}$  is a pseudovariety in  $\mathcal{A}$*
- *The objects of  $\mathcal{B}$  satisfy the affine descending chain condition.*

In general in this thesis we will assume that, for some theory containing the theory of groups, we have a category  $\mathcal{A}$  of algebras and a subcategory  $\mathcal{B}$  of finite algebras as defined above.

If  $\mathcal{A}$  is the category **Set** then the finite sets satisfy the conditions of Definition 2.3.3 so we can take  $\mathcal{B}$  to be the full subcategory of finite sets. Similarly if  $\mathcal{A}$  is the category **Vect** the finite dimensional vector spaces satisfy the conditions of 2.3.3 so we can take  $\mathcal{B}$  to be the full subcategory of finite dimensional vector spaces.

For convenience we use a different condition to the affine descending chain condition. We define this below and prove that it is equivalent to the affine descending chain condition.

**Definition 2.3.4.** *An algebra  $A$  satisfies the **finite sub-intersection property** if, given an indexing set  $\lambda$  and a family  $(S_i)_{i \in \lambda}$  of affine subsets of  $A$ , there is a finite subset  $I \subset \lambda$  such that*

$$\bigcap_{i \in \lambda} S_i = \bigcap_{i \in I} S_i.$$

**Proposition 2.3.5.** *An algebra satisfies the affine descending chain condition if and only if it satisfies the finite sub-intersection condition.*

*Proof.* Suppose we have an algebra  $A$  that satisfies the affine descending chain condition. Suppose also that we have an indexing set  $\lambda$  and a family  $(S_i)_{i \in \lambda}$  of affine subsets of  $A$ . Consider the affine subsets of  $A$  defined by

$$T_i = \bigcap_{0 \leq j \leq i} V_j.$$

This will be a descending chain, so for some  $n$ , we have  $T_i = T_n$  for all  $i \geq n$ . However this gives

$$\bigcap_{i \in \lambda} S_i = \bigcap_{i \in \lambda} T_i = T_n = \bigcap_{1 \leq i \leq n} S_i.$$

So  $A$  satisfies the finite sub-intersection property.

In the other direction suppose that the algebra  $A$  satisfies the finite sub-intersection property. Also, suppose that for some indexing set  $\gamma$  we have a descending chain  $(R_i)_{i \in \gamma}$  of affine subsets

of  $A$ . As  $A$  satisfies the finite sub-intersection property, there will be some finite subset  $I$  of  $\gamma$  with

$$\bigcap_{i \in \gamma} R_i = \bigcap_{i \in I} R_i.$$

Let  $j$  be the largest element in  $I$ . As the family  $(R_i)_{i \in \gamma}$  is a descending chain, we then have

$$\bigcap_{i \in \gamma} R_i = R_j.$$

Suppose we have a  $k \in \gamma$  with  $k \geq j$ . We then have  $R_k \subset R_j$ . However, we also have

$$R_j \cap R_k = \bigcap_{i \in \gamma} R_i \cap R_k = \bigcap_{i \in \gamma} R_i = R_j.$$

which gives  $R_j \subset R_k$ , which gives  $R_k = R_j$ . So  $A$  satisfies the affine descending chain condition.  $\square$

In this thesis we will primarily be interested in the affine cofinite subsets defined below.

**Definition 2.3.6.** *A subset  $S$  of an algebra  $A$  is **affine cofinite** if there exist a morphism  $A \xrightarrow{f} B$  with  $B$  finite and an element  $b \in B$  with  $S = f^{-1}(b)$ .*

In **Set** every subset will be affine cofinite. In **Vect** the affine cofinite subsets will be the finite codimensional affine subsets. Lemmas 2.2.3 and 2.2.4 can easily be restricted to the affine cofinite case to show that pre-images and intersections of affine cofinite subsets are affine cofinite.

Note that the empty subset is always affine cofinite. To see this, suppose we have an algebra  $A$ . Consider the morphism from  $A$  to the terminal algebra composed with any morphism from the terminal algebra to any finite algebra (which is not isomorphic to the terminal algebra). This morphism will have empty fibres in  $A$ , so the empty subset is affine cofinite.

## 2.4 Affine Images

In this section we will show that the image of an affine cofinite set under a surjective morphism is itself affine cofinite. This may seem an esoteric fact but it will prove pivotal later. We will be relying on the assumption that our theory contains the theory of groups repeatedly.

The following five lemmas will prove this result. For convenience, for the rest of this section we suppose that we have a surjective morphism  $A \xrightarrow{f} B$ , a morphism  $A \xrightarrow{h} C$  with  $C$  finite and a non-empty affine cofinite subset  $S$  of  $A$  such that  $S$  is equal to  $h^{-1}(1_C)$ .

The first thing we will do is define a relation  $\cong_S$  on  $B$  and show that it is a congruence.

**Definition 2.4.1.** *We define the relation  $\cong_S$  on  $B$  as follows. Given points  $b$  and  $b'$  in  $B$  if there are points  $a$  and  $a'$  in  $A$  with  $f(a) = b$  and  $f(a') = b'$  such that  $a$  and  $a'$  are in the same coset of  $S$  we then have  $b \cong_S b'$ .*

**Lemma 2.4.2.** *The relation  $\cong_S$  is an equivalence relation.*

*Proof.* The relation  $\cong_S$  is certainly reflexive and symmetric, we will show that it is transitive.

Suppose we have elements  $b, b'$  and  $b''$  in  $B$  with  $b \cong_S b'$  and  $b' \cong_S b''$ . So we have elements  $a_1, a_2, a_3$  and  $a_4$  of  $A$  with  $f(a_1) = b, f(a_2) = b', f(a_3) = b'$  and  $f(a_4) = b''$  such that  $a_1 \cdot S = a_2 \cdot S$  and  $a_3 \cdot S = a_4 \cdot S$ . Consider the elements  $a_1$  and  $a_2 \cdot a_3^{-1} \cdot a_4$  of  $A$ , we have

$$f(a_1) = b$$

and

$$f(a_2 \cdot a_3^{-1} \cdot a_4) = b' \cdot b'^{-1} \cdot b'' = b''$$

with

$$a_1 \cdot S = a_2 \cdot S = a_2 \cdot a_3^{-1} \cdot a_3 \cdot S = a_2 \cdot a_3^{-1} \cdot a_4 \cdot S.$$

So we have  $b \cong_S b''$ . Which means that  $\cong_S$  is an equivalence relation.  $\square$

We can use the following lemma to give a simpler formulation of the relation  $\cong_S$ .

**Lemma 2.4.3.** *Given points  $a$  and  $a'$  in  $A$ ,  $a$  and  $a'$  are in the same coset of  $S$  if and only if  $h(a) = h(a')$ .*

*Proof.* Suppose that  $a$  and  $a'$  are in the same coset of  $S$ , that is  $a \cdot S = a' \cdot S$ . We must then have  $a = a' \cdot s$  for some  $s \in S$ . So we have

$$h(a) = h(a' \cdot s) = h(a') \cdot h(s) = h(a').$$

In the other direction suppose we have  $h(a) = h(a')$ . By Lemma 2.2.5 we have

$$a \cdot h^{-1}(1) = h^{-1}(h(a)) = h^{-1}(h(a')) = a' \cdot h^{-1}(1).$$

So  $a \cdot S = a' \cdot S$ .  $\square$

We now show that the equivalence relation we have defined commutes with the operations of our algebra.

**Lemma 2.4.4.** *The relation  $\cong_S$  is a congruence.*

*Proof.* Suppose that  $\theta$  is an  $n$ -ary operation for the algebra  $B$  and that  $(b_i)_{1 \leq i \leq n}$  and  $(b'_i)_{1 \leq i \leq n}$  are families of elements of  $B$  with  $b_i \cong_S b'_i$  for  $1 \leq i \leq n$ . By the definition of  $\cong_S$  and Lemma 2.4.3, for  $1 \leq i \leq n$  we can choose an  $a_i$  and  $a'_i$  in  $A$  with  $f(a_i) = b_i$  and  $f(a'_i) = b'_i$  such that  $h(a_i) = h(a'_i)$ .

Now  $h$  is a morphism so will commute with the operation  $\theta$ , so we have

$$h(\theta(a_1, \dots, a_n)) = \theta(h(a_1), \dots, h(a_n)) = \theta(h(a'_1), \dots, h(a'_n)) = h(\theta(a'_1, \dots, a'_n)).$$

Again using the definition of  $\cong_S$  and Lemma 2.4.3, we have

$$f(\theta(a_1, \dots, a_n)) \cong_S f(\theta(a'_1, \dots, a'_n)).$$

So, as  $f$  is a morphism, we have

$$\theta(f(a_1), \dots, f(a_n)) = f(\theta(a_1, \dots, a_n)) \cong_S f(\theta(a'_1, \dots, a'_n)) = \theta(f(a'_1), \dots, f(a'_n))$$

However for  $1 \leq i \leq n$  we have  $f(a_i) = b_i$  and  $f(a'_i) = b'_i$  which gives

$$\theta(b_1, \dots, b_n) \cong_S \theta(b'_1, \dots, b'_n)$$

so our relation  $\cong_S$  is a congruence.  $\square$

We can now use the relation  $\cong_S$  to show that set  $f(S)$  is affine cofinite.

**Lemma 2.4.5.** *There is a morphism  $B \xrightarrow{p} D$ , with  $D$  finite, such that  $f(S) = p^{-1}(1_D)$ .*

*Proof.* By Lemma 2.4.4 the relation  $\cong_S$  is a congruence on  $B$  so defines a morphism  $B \xrightarrow{p} B/\cong_S$  such that for any  $b$  and  $b'$  in  $B$  we have  $b \cong_S b'$  if and only if  $p(b) = p(b')$ . We claim that  $f(S)$  is in fact equal to  $p^{-1}(1_D)$ . First suppose we have  $b \in f(S)$ , so we have  $b = f(s)$  for some  $s \in S$ . As  $f(1_A) = 1_B$  and  $sS = S = 1_AS$  we have  $b \cong_S 1_B$ . However, this means we have  $p(b) = p(1_B)$  which gives  $b \in p^{-1}(p(1_B)) = p^{-1}(1_D)$ . So we have  $f(S) \subset p^{-1}(1_D)$ .

In the other direction suppose we have  $b \in p^{-1}(1_D)$ , this gives  $b \cong_S 1_B$ . Then, by the definition of  $\cong_S$ , there are points  $a$  and  $a'$  in  $A$  with  $f(a) = b$  and  $f(a') = 1_B$  such that  $a$  and  $a'$  are in the same coset of  $S$ , that is  $a \cdot S = a' \cdot S$ .

So we have

$$b \cdot f(S) = f(a) \cdot f(S) = f(a \cdot S) = f(a' \cdot S) = f(a') \cdot f(S) = f(S).$$

As we have  $1_A \in S$  and  $f(1_A) = 1_B$  we have  $1_B \in f(S)$ , so in particular the above gives,  $b = b \cdot 1_B \in f(S)$ . So we have  $p^{-1}(1_D) \subset f(S)$ , which gives  $f(S) = p^{-1}(1_D)$  so  $f(S)$  is an affine set. As  $B/\cong_S$  is a quotient of the finite algebra  $B$ , it too is finite, so  $f(S)$  is affine cofinite.  $\square$

We can generalise the previous result to all affine cofinite subsets of  $A$ , not just those of the form  $h^{-1}(1_C)$  for some  $A \xrightarrow{h} C$  with  $C$  finite.

**Proposition 2.4.6.** *Given an affine cofinite subset  $R$  of  $A$ , then  $f(R)$  is an affine cofinite subset of  $B$ .*

*Proof.* If  $R$  is the empty set then  $f(R)$  is also empty so it too is affine cofinite. Assume then that we have  $R \neq \emptyset$ . As  $R$  is affine cofinite, there is a morphism  $A \xrightarrow{k} D$  with  $D$  in  $\mathcal{B}$  and an element  $d \in D$  with  $R = k^{-1}(d)$ . Let  $r$  be an element of  $R$ , by Lemma 2.2.5 we have  $R = rk^{-1}(1_D)$ . So, using Lemma 2.2.5 again, we have

$$f(R) = f(rk^{-1}(1_D)) = f(r)f(k^{-1}(1_D)).$$

However by Lemma 2.4.5 we have that  $f(k^{-1}(1_D))$  is affine cofinite. So, by Lemma 2.2.5, we have that  $f(r)f(k^{-1}(1_D))$  is also affine cofinite, which gives that  $f(R)$  is affine cofinite.  $\square$

## Chapter 3

# Ultrafilters

In this chapter we begin by restating the definition and basic properties of ordinary ultrafilters on sets. We then define the notion of an ultrafilter on an algebra and show that many of the properties of ordinary ultrafilters generalise to ultrafilters on algebras. Ultrafilters on sets are treated in detail in [4].

### 3.1 Ultrafilters on Sets

First some completely standard definitions and results.

**Definition 3.1.1.** *Given a set  $S$  and an indexing set  $\lambda$  the family  $(S_i)_{i \in \lambda}$  of subsets of  $S$  has the **finite intersection property** if for any finite subset  $I \subset \lambda$  we have*

$$\bigcap_{i \in I} S_i \neq \emptyset.$$

**Definition 3.1.2.** *A **filter**  $\mathcal{F}$  on a set  $S$  is a family of subsets of  $S$  satisfying*

- 1. The subset  $S$  is a member of the family  $\mathcal{F}$*
- 2. If  $U$  and  $V$  are in  $\mathcal{F}$  then we have  $U \cap V \in \mathcal{F}$*
- 3. If  $U$  is in  $\mathcal{F}$  and  $W$  is a subset of  $S$  with  $U \subset W$  we have  $W \in \mathcal{F}$ .*

By induction filters are closed under all finite intersection. As it is closed under supersets, if a filter on  $S$  contains the empty set,  $\emptyset$ , it will contain every subset of  $S$ . Such a filter is referred to as the trivial filter on  $S$ . By reference to the following definition we will exclude this case.

**Definition 3.1.3.** *A filter is **proper** if it does not contain the empty subset.*

A corollary of this definition is that there are no proper filters on the empty set. Given a subset  $S$  of some set  $T$  we write  $S^c$  for the complement of  $S$  in  $T$ . We now define certain distinguished filters called ultrafilters. Ultrafilters are useful in set theory, model theory and topology.

**Definition 3.1.4.** *An **ultrafilter**  $\mathcal{U}$  on a set  $S$  is a filter such that for any  $T \subset S$  either  $T \in \mathcal{U}$  or  $T^c \in \mathcal{U}$ , but not both.*

Suppose that  $\mathcal{U}$  is an ultrafilter. If we have  $\emptyset \in \mathcal{U}$  then, for any  $T \subset S$  we would have  $T \in \mathcal{U}$  and  $T^c \in \mathcal{U}$ , contradicting the definition of ultrafilter. So ultrafilters are proper filters. This gives that there are no ultrafilters on the empty set. The simplest ultrafilters are the principal ultrafilters.

**Definition 3.1.5.** Give an element  $s$  of a set  $S$  the **principal ultrafilter**  $\mathcal{U}_s$  at  $s$  is the family of subsets containing the point  $s$ .

Indeed for finite  $S$ , as the next lemma shows, principal ultrafilters are the only ultrafilters. We will need to consider infinite  $S$  to find more interesting examples.

**Lemma 3.1.6.** Every ultrafilter on a finite set is principal.

*Proof.* Suppose we have an ultrafilter  $\mathcal{U}$  on a finite set  $S$ . For each  $s \in S$  we have a pair of sets  $\{s\}$  and  $S \setminus \{s\}$ . If for all  $s \in S$  we had  $S \setminus \{s\} \in \mathcal{U}$  then, as there are finitely many such sets, we would have

$$\emptyset = \bigcap_{s \in S} S \setminus \{s\} \in \mathcal{U}.$$

This contradicts the properness of  $\mathcal{U}$ . So for some  $s \in S$  we have  $\{s\} \in \mathcal{U}$ . As  $\mathcal{U}$  is closed under taking supersets we then have  $\mathcal{U}_s \subset \mathcal{U}$ . Indeed if we had some set  $T \in \mathcal{U}$  with  $s \notin T$  then, as  $\{s\} \cap T = \emptyset$ , this would contradict the properness of  $\mathcal{U}$ . So we have  $\mathcal{U} = \mathcal{U}_s$   $\square$

Whereas principal ultrafilters provide a ready supply of ultrafilters, non-principal ultrafilters cannot be explicitly constructed. However, we will see that any family of sets with the finite intersection property is contained in an ultrafilter. Ultrafilters constructed in this manner may or may not be principal. First we show that any family of sets with the finite intersection property is contained in a proper filter.

**Lemma 3.1.7.** Every family of subsets of some set  $S$  with the finite intersection property can be extended to a proper filter on  $S$ .

*Proof.* Suppose we have a set  $S$  and, for some indexing set  $\lambda$ , a family  $(T_i)_{i \in \lambda}$  of subsets of  $S$  with the finite intersection property. Consider the set

$$\mathcal{F} = \{U \subset S \mid \exists I \subset \lambda, I \text{ finite, with } \bigcap_{i \in I} T_i \subset U\}$$

By construction this set  $\mathcal{F}$  is a filter. As the family of sets  $(T_i)_{i \in \lambda}$  has the finite intersection property we have  $\emptyset \notin \mathcal{F}$ . So this set  $\mathcal{F}$  is a proper filter containing the original family of sets.  $\square$

We say that a filter on a set is maximal if it is not contained in a bigger proper filter. We now see that every filter is contained in a maximal filter. As we use Zorn's lemma this result requires the axiom of choice.

**Lemma 3.1.8.** Every filter on a set can be extended to a maximal filter on that set.

*Proof.* Suppose  $\mathcal{T}$  is a filter on the set  $S$ . Consider the set of filters on  $S$  that contain the filter  $\mathcal{T}$ . This set is partially ordered by inclusion. Suppose also that for some indexing set  $\lambda$  we have

a totally ordered subset  $(\mathcal{F}_i)_{i \in \lambda}$  of this set. Then the set

$$\mathcal{F} = \bigcup_{i \in \lambda} \mathcal{F}_i$$

will also be a filter with  $\mathcal{T} \subset \mathcal{F}$  and we will have  $\mathcal{F}_i \subset \mathcal{F}$  for all  $i \in \lambda$ . So the chain  $(\mathcal{F}_i)_{i \in \lambda}$  will have an upper bound.

So, by Zorn's Lemma, the set of all filters containing  $\mathcal{T}$  has a maximal element, we denote this filter  $\mathcal{U}$ . Indeed  $\mathcal{U}$  will be maximal in the set of all filters on  $S$ , suppose we have a filter  $\mathcal{F}'$  with  $\mathcal{U} \subset \mathcal{F}'$ , then we would have  $\mathcal{T} \subset \mathcal{F}'$ , however  $\mathcal{U}$  is maximal among filter containing  $\mathcal{T}$  so we have  $\mathcal{U} = \mathcal{F}'$ .  $\square$

With the previous two lemmas we can now prove that ultrafilters are the same thing as maximal proper filters.

**Proposition 3.1.9.** *A family of subsets of a set is an ultrafilter if and only if it is a maximal proper filter.*

*Proof.* Suppose that  $\mathcal{F}$  is a maximal proper filter on the set  $S$ . So for some indexing set  $\lambda$  we have  $\mathcal{F} = (S_i)_{i \in \lambda}$  where the  $S_i$  are subsets of  $S$ . Let  $T$  be a subset of  $S$ , we will show that either  $T$  or  $T^c$  is in  $\mathcal{F}$ .

Suppose, that there is an  $i \in \lambda$  such that  $T \cap S_i = \emptyset$  and a  $j \in \lambda$  such that  $T^c \cap S_j = \emptyset$ . Then we would have  $S_j \subset T$  and  $S_i \cap T = \emptyset$ . However this gives,  $S_i \cap S_j = \emptyset$ , a contradiction as  $S_i$  and  $S_j$  are elements of the proper filter  $\mathcal{F}$ . So, either we have  $T \cap S_i \neq \emptyset$  for all  $i \in \lambda$  or we have  $T^c \cap S_j \neq \emptyset$  for all  $j \in \lambda$ . That is, either  $T$  or  $T^c$  has non-empty intersection with every element of  $\mathcal{F}$ .

Without loss of generality let's suppose it is  $T$  that has this property. By Lemma 3.1.7 we can then extend  $\{T\} \cup \mathcal{F}$  to a filter, however  $\mathcal{F}$  is maximal, so we must have  $T \in \mathcal{F}$ . As  $\mathcal{F}$  is proper, we cannot have  $T^c \in \mathcal{F}$ , so  $\mathcal{F}$  is an ultrafilter.

Now suppose that  $\mathcal{U}$  is an ultrafilter on  $S$ , by definition it is a filter. As we have already observed, every ultrafilter is a proper filter, we will show that it is a maximal. Let  $\mathcal{V}$  be a proper filter on  $S$  with  $\mathcal{U} \subset \mathcal{V}$ . If we have some  $T \in \mathcal{V}$  such that  $T \notin \mathcal{U}$ , then we have  $T^c \in \mathcal{U}$ , however this gives  $T^c \in \mathcal{V}$  which contradicts the properness of  $\mathcal{V}$ . So we have  $\mathcal{U} \subset \mathcal{V}$ , that is,  $\mathcal{U}$  is a maximal proper filter.  $\square$

Using Lemmas 3.1.7 and 3.1.8 we can extend any family of sets with the finite intersection property to a maximal filter, and hence, to an ultrafilter. So we have only two ways to construct ultrafilters, either as the principal ultrafilter at a point or as the maximal filter containing some family of sets with the finite intersection property.

We can use the second of these methods to construct an example of a non-principal ultrafilter on an infinite set. We say a subset of an infinite set is cofinite if its complement is finite. Given an infinite set  $S$  consider the family of cofinite subsets of  $S$ . This family will have the finite intersection property so we can extend it to an ultrafilter  $\mathcal{U}$ . For any  $s \in S$  we have that  $S \setminus s$  is cofinite so  $S \setminus s \in \mathcal{U}$ , that is  $\{s\} \notin \mathcal{U}$ , so  $\mathcal{U}$  is non-principal.



## 3.2 Ultrafilters on Algebras

We now define ultrafilters on algebras as a generalisation of ultrafilters on sets. We also show that these ultrafilters satisfy many of the same properties of ultrafilters on sets.

### 3.2.1 Defining Ultrafilters

Galvin and Horn prove a slightly more general version of the following result in [8].

**Lemma 3.2.1.** *A family of subsets  $\mathcal{U}$  of a set  $S$  is an ultrafilter on  $S$  if and only if, for any partition of  $S$  into finitely many disjoint subsets, exactly one element of the partition is in  $\mathcal{U}$ .*

*Proof.* Suppose that  $\mathcal{U}$  is an ultrafilter on the set  $S$ . Suppose also that for some finite  $n$  we have a partition of  $S$  into the subsets  $(S_i)_{1 \leq i \leq n}$ . As these subsets are disjoint at most one of them can be an element of  $\mathcal{U}$ . Suppose that none of them are, that is, for  $1 \leq i \leq n$ , we have  $S_i \notin \mathcal{U}$ . Then we have  $S_i^c \in \mathcal{U}$  for  $1 \leq i \leq n$ . However this gives

$$\emptyset = \bigcap_{1 \leq i \leq n} S_i^c \in \mathcal{U}$$

which contradicts the properness of  $\mathcal{U}$ . So for some unique  $1 \leq j \leq n$  we have  $S_j \in \mathcal{U}$ .

In the other direction, suppose that  $\mathcal{U}$  is a family of sets satisfying the above condition. We can partition  $S$  as  $S \sqcup \emptyset \sqcup \emptyset$ , as exactly one of these sets is in  $\mathcal{U}$  we have  $\emptyset \notin \mathcal{U}$ . Suppose we have subset  $T$  of  $S$ , then  $T$  and  $T^c$  form a partition of  $S$  so exactly one of these sets is in  $\mathcal{U}$ .

Suppose we have subsets  $T$  and  $T'$  of  $S$  with  $T \subset T'$  and  $T \in \mathcal{U}$ . Consider the partition

$$S = T \sqcup (T' \setminus T) \sqcup (S \setminus T').$$

Exactly one of these sets is in  $\mathcal{U}$ , so, as we have  $T \in \mathcal{U}$ , we must have  $S \setminus T' \notin \mathcal{U}$ , that is,  $T' \in \mathcal{U}$ . So  $\mathcal{U}$  is upwards closed.

Suppose we have subsets  $T$  and  $T'$  of  $S$  with  $T \in \mathcal{U}$  and  $T' \in \mathcal{U}$ . If we had  $T \cap T' = \emptyset$  then  $T \subset S \setminus T'$  so by upwards closure we have  $S \setminus T' \in \mathcal{U}$  that is,  $T'^c \in \mathcal{U}$ , a contradiction. We can partition  $S$  as

$$S = (T \cap T') \sqcup (T \setminus T') \sqcup (S \setminus T).$$

Exactly one of these sets will be in  $\mathcal{U}$ , however as  $T$  and  $T'$  are in  $\mathcal{U}$  we can't have  $T \setminus T' \in \mathcal{U}$  or  $S \setminus T \in \mathcal{U}$ , so we have  $T \cap T' \in \mathcal{U}$ . So  $\mathcal{U}$  is an ultrafilter on  $S$ .  $\square$

So, using this lemma, we could alternatively define ultrafilters in terms of partitions as opposed to subsets. However, a partition of a set  $S$  into  $n$  segments is equivalent to a morphism from  $S$  into the  $n$  element set. So, as in [13], with this observation and Lemma 3.2.1 we can make the following equivalent definition of an ultrafilter on a set.

**Definition 3.2.2.** *An **ultrafilter**  $\mathcal{U}$  on a set  $S$  is a family of subsets of  $S$  such that, for any function  $S \xrightarrow{f} n$  with  $n$  finite, there is exactly one  $m$  in  $n$  such that  $f^{-1}(m) \in \mathcal{U}$ .*

This is the definition of ultrafilter on a set that we will generalise to algebras. We might be tempted to define an ultrafilter on an algebra  $A$  as a family  $\mathcal{U}$  of affine cofinite subsets of  $A$  such that for any morphism  $A \xrightarrow{f} B$  with  $B$  finite there is a unique  $b$  in  $B$  such that  $f^{-1}(b) \in \mathcal{U}$ .

Unfortunately this simple generalisation will not work as one would hope. To see this consider the vector space  $\mathbb{R}^2$ , let  $\mathcal{U}$  be the collection of lines through the origin, some point distinct from the origin and the whole of  $\mathbb{R}^2$ . This will satisfy the proposed definition above, however it is not at all like an ultrafilter on a set, it is not closed under intersections or supersets and despite being an ultrafilter on a finite dimensional vector space is not principal. We are obliged to use a slightly more complicated definition to exclude possibilities such as this.

**Definition 3.2.3.** An **ultrafilter**  $\mathcal{U}$  on an algebra  $A$  is a collection of affine cofinite subsets of  $A$  with the property that for any  $A \xrightarrow{f} B$  with  $B$  finite there is a unique  $b \in B$  with  $f^{-1}(b) \in \mathcal{U}$ , and if there is an affine cofinite subset  $S \subset B$ , with  $f^{-1}(S) \in \mathcal{U}$  then  $b$  is an element of  $S$ .

In the **Set** case, the second condition of this definition can be derived from Definition 3.2.1. To see this, suppose we have some morphism  $A \xrightarrow{f} B$ , with  $B$  finite, and there is an element  $b$  in  $B$  and a subset  $S$  of  $B$  with  $f^{-1}(b) \in \mathcal{U}$  and  $f^{-1}(S) \in \mathcal{U}$ . Assume we have  $b \notin S$  and let  $T$  be the subset  $A \setminus (f^{-1}(b) \cup f^{-1}(S))$ , then we can disjointly partition  $A$  as

$$A = f^{-1}(b) \cup f^{-1}(S) \cup T.$$

However both  $f^{-1}(b)$  and  $f^{-1}(S)$  are elements of  $\mathcal{U}$ , contradicting the fact that  $\mathcal{U}$  is an ultrafilter. So we must have  $b \in S$ . In particular this means that in the set case Definition 3.2.3 reduces to Definition 3.2.1. In the affine case we cannot construct arbitrary partitions as we can for sets so we cannot use the above argument.

Henceforth, we follow the integration notation of [13], given an ultrafilter  $\mathcal{U}$  on an algebra  $A$  and a morphism  $A \xrightarrow{f} B$  with  $B$  finite, we write  $\int f d\mathcal{U}$  for the unique element  $b \in B$  with  $f^{-1}(b) \in \mathcal{U}$ .

The justification for this in the **Set** case is as follows. Given an ultrafilter  $\mathcal{U}$  on a set  $S$ , we have a map  $\int - d\mathcal{U}$  that sends a function  $S \xrightarrow{f} n$  to an element  $\int f d\mathcal{U}$  of  $n$ . Given a morphism  $S \xrightarrow{f} n$  with constant value  $m$  we have  $\int f d\mathcal{U} = m$ . Also if we have two functions  $S \xrightarrow{f} n$  and  $S \xrightarrow{g} n$  with  $\{s \in S | f(s) = g(s)\} \in \mathcal{U}$  then we have  $\int f d\mathcal{U} = \int g d\mathcal{U}$ . Indeed  $\int - d\mathcal{U}$  is the unique map with these properties, so it is in some way analogous to the ordinary notion of integration. For the details see proposition 3.2 in [13].

Just as integration in the usual sense is linear, we have the following result.

**Lemma 3.2.4.** If  $\mathcal{U}$  is an ultrafilter on  $A$ , then for any morphism  $A \xrightarrow{f} B$  with  $B$  finite and any morphism  $B \xrightarrow{h} C$  with  $C$  finite, we have

$$h(\int f d\mathcal{U}) = \int hf d\mathcal{U}.$$

*Proof.* To see this, observe that as  $f^{-1}(S) \in \mathcal{U}$  gives  $\int f d\mathcal{U} \in S$  then for any  $B \xrightarrow{h} C$  with  $C$  finite, as

$$f^{-1}(h^{-1}(\int hf d\mathcal{U})) = (hf)^{-1}(\int hf d\mathcal{U}) \in \mathcal{U}$$

we will have

$$\int f d\mathcal{U} \in h^{-1}(\int hf d\mathcal{U})$$

which gives

$$h(\int f d\mathcal{U}) = \int hf d\mathcal{U}.$$

□

Indeed, we can use this condition to rephrase the definition of ultrafilter.

**Lemma 3.2.5.** *Let  $\mathcal{V}$  be a collection of affine cofinite subsets of  $A$  with the property that for any morphism  $A \xrightarrow{f} B$  with  $B$  finite there will be a unique element  $\int f d\mathcal{U}$  of  $B$  with  $f^{-1}(\int f d\mathcal{U})$  in  $\mathcal{U}$  and for any  $B \xrightarrow{h} C$  with  $C$  finite, we have*

$$h(\int f d\mathcal{U}) = \int (hf d\mathcal{U}).$$

*Then  $\mathcal{V}$  will be an ultrafilter on  $A$ .*

*Proof.* To show that  $\mathcal{V}$  is an ultrafilter we just need to prove that if we have an affine cofinite subset  $S \subset B$ , with  $f^{-1}(S) \in \mathcal{U}$  then  $\int f d\mathcal{U} \in S$ . Suppose we have such an  $S$ , then we will also have some  $B \xrightarrow{k} D$  with  $D$  finite and  $d \in D$  with  $S = k^{-1}(d)$ .

So we have

$$f^{-1}(k^{-1}(d)) = f^{-1}(S) \in \mathcal{U}$$

so, by uniqueness, we have  $\int kf d\mathcal{U} = d$ , which gives,  $k(\int f d\mathcal{U}) = d$ , that is

$$\int f d\mathcal{U} \in S$$

and so  $\mathcal{V}$  is an ultrafilter. □

We can show that ultrafilters actually satisfy a slightly stronger condition than that given in the definition.

**Lemma 3.2.6.** *If  $\mathcal{U}$  is an ultrafilter on  $A$ , we have a morphism  $A \xrightarrow{f} B$  with  $B$  finite and  $S \subset B$  is affine cofinite then  $\int f d\mathcal{U} \in S$  if and only if  $f^{-1}(S) \in \mathcal{U}$ .*

*Proof.* From the definition of ultrafilter we already have that if  $f^{-1}(S) \in \mathcal{U}$  then  $\int f d\mathcal{U} \in S$ . In the other direction assume that we have  $\int f d\mathcal{U} \in S$ . As  $S$  is affine cofinite we have a morphism  $B \xrightarrow{h} C$  with  $C$  finite and a  $c \in C$  with  $S = h^{-1}(c)$ .

As  $\int f d\mathcal{U} \in S$  we have  $\int f d\mathcal{U} \in h^{-1}(c)$  so

$$h(\int f d\mathcal{U}) = c$$

which, by Lemma 3.2.4, gives

$$\int hf d\mathcal{U} = c$$

and so we have

$$f^{-1}(S) = f^{-1}(h^{-1}(c)) = (hf)^{-1}(\int hf d\mathcal{U}) \in \mathcal{U}$$

that is,  $f^{-1}(S) \in \mathcal{U}$  □

As with ultrafilters on sets we can define principal ultrafilters on algebras.

**Definition 3.2.7.** *We say that an ultrafilter  $\mathcal{U}$  on an algebra  $A$  is **principal** if there exists a point  $p \in A$  such that for every morphism  $A \xrightarrow{f} B$  with  $B$  finite we have*

$$\int f d\mathcal{U} = f(p).$$

If  $\mathcal{U}$  and  $p$  are as above, we say that  $\mathcal{U}$  is principal at  $p$ . As we see in the following lemma this notion of principal ultrafilter is a direct generalisation of the notion of principal ultrafilter on a set.

**Lemma 3.2.8.** *An ultrafilter  $\mathcal{U}$  on an algebra  $A$  is principal at  $p$  if and only if for any affine cofinite subset  $S$  of  $A$  we have*

$$p \in S \Leftrightarrow S \in \mathcal{U}.$$

*Proof.* Suppose we have an affine cofinite subset  $S$  of  $A$  and a morphism  $A \xrightarrow{f} B$  with  $B$  finite and a  $b \in B$  such that  $S = f^{-1}(b)$ . Then we have

$$S \in \mathcal{U} \Leftrightarrow \int f d\mathcal{U} = b$$

and

$$p \in S \Leftrightarrow f(p) = b.$$

So the condition that for all affine cofinite subsets  $S$  of  $A$  we have

$$p \in S \Leftrightarrow S \in \mathcal{U}$$

is equivalent to the condition that for all  $A \xrightarrow{f} B$  and  $b \in B$  we have

$$\int f d\mathcal{U} = b \Leftrightarrow f(p) = b.$$

That is,

$$\int f d\mathcal{U} = f(p).$$

□

In the next lemma we see that, as in the set case, the only ultrafilters on finite algebras will be the principal ones.

**Lemma 3.2.9.** *If  $\mathcal{U}$  is an ultrafilter on a finite algebra  $B$ , then  $\mathcal{U}$  is principal.*

*Proof.* Suppose we have an ultrafilter  $\mathcal{U}$  on the finite algebra  $B$  and let  $id_B$  be the identity morphism on  $B$ . Given a morphism  $B \xrightarrow{h} C$  with  $C$  finite, by Lemma 3.2.4 we have

$$\int h d\mathcal{U} = h(\int id_B d\mathcal{U}).$$

So, we have that  $\mathcal{U}$  is principal at  $\int id_B d\mathcal{U}$ .

□

### 3.2.2 The functor $\mathcal{U}$

We can define an endofunctor on the category of sets that sends a set  $S$  to the set of ultrafilters on  $S$  and sends a morphism  $S \xrightarrow{f} T$  between sets to the ultrafilter pushforward morphism which we define below.

Similarly, if  $\mathcal{A}$  is our category of algebras, we can define an ultrafilter functor from  $\mathcal{A}$  to the category of sets. First we must define the pushforward of an ultrafilter. This is a direct generalisation of the set case.

**Definition 3.2.10.** Given algebras  $A$  and  $B$  in  $\mathcal{A}$ , a morphism  $A \xrightarrow{g} B$  and an ultrafilter  $\mathcal{U}$  on  $A$ , the **pushforward**  $g_*(\mathcal{U})$  is the set of affine cofinite subsets of  $B$  given by,  $S \in g_*(\mathcal{U})$  if and only if  $g^{-1}(S) \in \mathcal{U}$ .

We must show that with this definition the pushforward of an ultrafilter is in fact an ultrafilter.

**Lemma 3.2.11.** Given algebras  $A$  and  $B$  in  $\mathcal{A}$ , a morphism  $A \xrightarrow{g} B$  and an ultrafilter  $\mathcal{U}$  on  $A$ , the pushforward  $g_*(\mathcal{U})$  will be an ultrafilter on  $B$ .

*Proof.* For each  $B \xrightarrow{f} C$  with  $C$  finite, as  $\mathcal{U}$  is an ultrafilter on  $A$ , there is a unique element  $\int f g d\mathcal{U}$  of  $C$  such that  $(fg)^{-1}(\int f g d\mathcal{U}) \in \mathcal{U}$ , however this gives

$$g^{-1}(f^{-1}(\int f g d\mathcal{U})) = (fg)^{-1}(\int f g d\mathcal{U}) \in \mathcal{U}$$

which, by definition of the pushforward, is equivalent to,  $f^{-1}(\int f g d\mathcal{U}) \in g_*(\mathcal{U})$ . So if we let

$$\int f d g_*(\mathcal{U}) = \int f g d\mathcal{U}$$

then we have

$$f^{-1}(\int f d g_*(\mathcal{U})) \in g_*(\mathcal{U}).$$

To see that  $\int f d g_*(\mathcal{U})$  is the unique element of  $C$  such that  $f^{-1}(\int f d g_*(\mathcal{U})) \in g_*(\mathcal{U})$ , suppose we have a  $b \in B$  with  $f^{-1}(b) \in g_*(\mathcal{U})$ , then we must have

$$g^{-1}(f^{-1}(b)) \in \mathcal{U} \Leftrightarrow b = \int f g d\mathcal{U} \Leftrightarrow b = \int f d g_*(\mathcal{U}).$$

If we have an affine cofinite  $S \subset C$ , with  $f^{-1}(S) \in g_*(\mathcal{U})$ , then  $g^{-1}(f^{-1}(S)) \in \mathcal{U}$  and so

$$\int f d g_*(\mathcal{U}) = \int f g d\mathcal{U} \in S$$

So  $g_*(\mathcal{U})$  is an ultrafilter. □

We can now define the ultrafilter functor from the category of algebras,  $\mathcal{A}$ , to **Set**.

**Definition 3.2.12.** For  $A$  an algebra we define  $\mathfrak{U}(A)$  to be the set of ultrafilters on  $A$  and given a morphism  $A \xrightarrow{f} B$ , for each  $\mathcal{U} \in \mathfrak{U}(A)$  we let the morphism  $\mathfrak{U}(A) \xrightarrow{\mathfrak{U}(f)} \mathfrak{U}(B)$  be defined by  $\mathfrak{U}(f)(\mathcal{U}) = f_*(\mathcal{U})$ .

**Lemma 3.2.13.**  $\mathfrak{U} : \mathcal{A} \rightarrow \mathbf{Set}$  is an functor.

*Proof.* Suppose  $\mathcal{U}$  is an ultrafilter on an algebra  $A$ . We easily have  $\mathfrak{U}(1_A)(\mathcal{U}) = (1_A)_*(\mathcal{U}) = \mathcal{U}$ , so  $\mathfrak{U}$  preserves identities. To show  $\mathfrak{U}$  preserves composition suppose we have morphisms  $A \xrightarrow{f} B \xrightarrow{g} C$  and  $\mathcal{U} \in \mathfrak{U}(A)$ . We have  $\mathfrak{U}(gf)(\mathcal{U}) = (gf)_*(\mathcal{U})$  and  $\mathfrak{U}(g)(\mathfrak{U}(f)(\mathcal{U})) = g_*(f_*(\mathcal{U}))$ .

However for  $X$  an affine cofinite subset of  $C$ , we have

$$X \in g_*(f_*(\mathcal{U})) \Leftrightarrow f^{-1}(g^{-1}(X)) \in \mathcal{U} \Leftrightarrow X \in (gf)_*(\mathcal{U})$$

so,  $\mathfrak{U}(g)(\mathfrak{U}(f)(\mathcal{U})) = \mathfrak{U}(gf)(\mathcal{U})$  which means  $\mathfrak{U}$  preserves composition, and so is a functor. □

### 3.2.3 Properties of Ultrafilters

We don't normally think of ultrafilters as they are defined in definition 3.2.1, instead we think of them as filters satisfying certain extra conditions. Now we will see that ultrafilters on algebras are actually filters as well. First we note the following lemma.

**Lemma 3.2.14.** *There are no ultrafilters on the empty algebra, whenever it exists.*

*Proof.* Supposing it exists, we let  $A$  be the empty algebra. The empty algebra is a subalgebra of every finite algebra, so as our finite algebras are a pseudovariety, the empty algebra is itself finite. Consider the morphism  $A \xrightarrow{f} A$ , as  $A$  is empty we cannot have a  $b \in A$  with  $f^{-1}(b) \in \mathcal{U}$ , so there are no ultrafilters on  $A$ .  $\square$

In the following two lemmas we show that an ultrafilter on an algebra  $A$  always contains the affine cofinite subset  $A$  and never contains the affine cofinite subset  $\emptyset$ . The later is a fact we will rely upon frequently in the sequel.

**Lemma 3.2.15.** *If  $\mathcal{U}$  is an ultrafilter on the algebra  $A$ , then we have  $A \in \mathcal{U}$ .*

*Proof.* Our finite algebras are a pseudovariety so they are closed under finite products so we have that the terminal algebra, the empty product, is finite. Consider the morphism  $f$  from  $A$  to the terminal algebra. Then we have

$$A = f^{-1}(\int f) \in \mathcal{U}$$

$\square$

**Lemma 3.2.16.** *For an ultrafilter  $\mathcal{U}$  on an algebra  $A$ ,  $\emptyset \notin \mathcal{U}$ .*

*Proof.* Assume  $\emptyset \in \mathcal{U}$ , there there is some  $A \xrightarrow{f} B$  with  $B$  finite such that  $f^{-1}(\int f) = \emptyset$ . This is equivalent to saying that  $\int f$  is not in the image of  $f$ . Which, by the definition of ultrafilter implies that

$$f^{-1}(\text{Im } f) \notin \mathcal{U}$$

however,  $f^{-1}(\text{Im } f) = A$  so this is equivalent to  $A \notin \mathcal{U}$ , which contradicts Lemma 3.2.15, so  $\emptyset \notin \mathcal{U}$ .  $\square$

We can define filters on algebras much as we did in the set case.

**Definition 3.2.17.** *Given an algebra  $A$ , a family of affine cofinite subsets of  $A$  is a **filter** if it is closed under finite intersections and affine cofinite supersets.*

Henceforth, when we use the term filter we will be referring to filters of affine cofinite subsets as defined above. With the following two lemmas we see that our ultrafilters on algebras are in fact filters.

**Lemma 3.2.18.** *Ultrafilters are closed under finite intersections.*

*Proof.* Let  $\mathcal{U}$  be an ultrafilter on an algebra  $A$ . By Lemma 3.2.15 the empty intersection,  $A$ , will be in  $\mathcal{U}$ . For binary intersections, let  $V$  and  $W$  be elements of  $\mathcal{U}$ , so there are morphisms  $A \xrightarrow{f} B$  and  $A \xrightarrow{g} C$  with  $B$  and  $C$  finite such that  $V = f^{-1}(\int f d\mathcal{U})$  and  $W = g^{-1}(\int g d\mathcal{U})$ .

We have

$$V \cap W = f^{-1}\left(\int f d\mathcal{U}\right) \cap g^{-1}\left(\int g d\mathcal{U}\right).$$

Now consider the product morphism induced by  $f$  and  $g$

$$\begin{array}{ccccc} & & A & & \\ & f \swarrow & \downarrow \langle f, g \rangle & \searrow g & \\ B & \xleftarrow{p_1} & B \times C & \xrightarrow{p_2} & C \end{array}$$

Given a point  $x$  in  $A$  we see that  $x \in f^{-1}(\int f d\mathcal{U}) \cap g^{-1}(\int g d\mathcal{U})$  is equivalent to  $f(x) = \int f d\mathcal{U}$  and  $g(x) = \int g d\mathcal{U}$ , that is,

$$\langle f, g \rangle(x) = \langle \int f d\mathcal{U}, \int g d\mathcal{U} \rangle$$

so this gives

$$f^{-1}\left(\int f d\mathcal{U}\right) \cap g^{-1}\left(\int g d\mathcal{U}\right) = \langle f, g \rangle^{-1}\left(\langle \int f d\mathcal{U}, \int g d\mathcal{U} \rangle\right).$$

This gives us

$$V \cap W = \langle f, g \rangle^{-1}\left(\langle \int f d\mathcal{U}, \int g d\mathcal{U} \rangle\right).$$

However, by Lemma 3.2.4 we will have

$$p_1 \int \langle f, g \rangle d\mathcal{U} = \int p_1 \langle f, g \rangle d\mathcal{U} = \int f d\mathcal{U}.$$

This gives

$$\langle \int f d\mathcal{U}, \int g d\mathcal{U} \rangle = \int \langle f, g \rangle d\mathcal{U}$$

so, putting this all together, we have

$$V \cap W = \langle f, g \rangle^{-1}\left(\int \langle f, g \rangle d\mathcal{U}\right) \in \mathcal{U}$$

so  $V \cap W \in \mathcal{U}$  and our ultrafilter  $\mathcal{U}$  is closed under finite intersections.  $\square$

In Lemma 3.2.16 we showed that for all ultrafilters  $\mathcal{U}$  we have  $\emptyset \notin \mathcal{U}$ . We use this fact now to prove that ultrafilters are closed under taking supersets.

**Lemma 3.2.19.** *Ultrafilters are closed under affine cofinite supersets.*

*Proof.* Let  $\mathcal{U}$  be an ultrafilter on an algebra  $A$ ,  $V$  be an element of  $\mathcal{U}$  and  $W$  be an affine cofinite subset of  $A$  with  $V \subset W$ , we will show  $W \in \mathcal{U}$ . Suppose we have morphisms  $A \xrightarrow{f} B$  and  $A \xrightarrow{g} C$  with  $B$  and  $C$  finite and a point  $c$  in  $C$  such that  $V = f^{-1}(\int f d\mathcal{U})$  and  $W = g^{-1}(c)$ .

As  $\mathcal{U}$  is an ultrafilter we have  $g^{-1}(\int g d\mathcal{U}) \in \mathcal{U}$  and, as  $\mathcal{U}$  is closed under intersections by Lemma 3.2.18, we have that

$$g^{-1}\left(\int g d\mathcal{U}\right) \cap f^{-1}\left(\int f d\mathcal{U}\right) \in \mathcal{U}$$

and, as  $\emptyset \notin \mathcal{U}$  by Lemma 3.2.16, we have  $g^{-1}(\int g d\mathcal{U}) \cap f^{-1}(\int f d\mathcal{U}) \neq \emptyset$ . But we also have

$$g^{-1}(\int g d\mathcal{U}) \cap f^{-1}(\int f d\mathcal{U}) \subset g^{-1}(\int g d\mathcal{U}) \cap g^{-1}(c)$$

so  $g^{-1}(\int g d\mathcal{U}) \cap g^{-1}(c) \neq \emptyset$ . This implies that  $c = \int g d\mathcal{U}$  and so

$$W = g^{-1}(c) = g^{-1}(\int g d\mathcal{U}) \in \mathcal{U}$$

So  $\mathcal{U}$  is closed under affine cofinite supersets.  $\square$

So, by Lemmas 3.2.18 and 3.2.19, we have seen that ultrafilters are filters on  $A$ . Indeed, by Lemma 3.2.16, ultrafilters are proper filters.

### 3.3 Maximal Filters

We know from Proposition 3.1.9 that an ultrafilter on a set  $S$  is the same thing as a maximal proper filter on  $S$ . We would like to prove an analogous result for ultrafilters on algebras, that is, that an ultrafilter on an algebra  $A$  is the same thing as a maximal proper filter on  $A$ .

As we have seen in the previous section every ultrafilter on an algebra  $A$  will be a proper filter on  $A$ . We will now see that every ultrafilter is a maximal proper filter on  $A$ . Proving the converse, that every maximal proper filter on  $A$  is slightly more involved.

**Lemma 3.3.1.** *Every ultrafilter on an algebra  $A$  is a maximal proper filter on  $A$ .*

*Proof.* Let  $\mathcal{U}$  be an ultrafilter on  $A$ . We have already seen that  $\mathcal{U}$  is a proper filter on  $A$ . We will show that it is maximal. Let  $\mathcal{V}$  be a proper filter on  $A$  with  $\mathcal{U} \subset \mathcal{V}$ . If we have  $W \in \mathcal{V}$ , then, as  $W$  is affine cofinite, there is a morphism  $A \xrightarrow{f} B$ , with  $B$  finite and a  $b \in B$  with  $f^{-1}(b) = W$ .

As  $\mathcal{U}$  is an ultrafilter we have  $f^{-1}(\int f d\mathcal{U}) \in \mathcal{U}$  which gives  $f^{-1}(\int f d\mathcal{U}) \in \mathcal{V}$ , so, as

$$f^{-1}(\int f d\mathcal{U}) \cap f^{-1}(b) \in \mathcal{V}$$

and  $\mathcal{V}$  is proper, we must have

$$f^{-1}(\int f d\mathcal{U}) \cap f^{-1}(b) \neq \emptyset$$

so  $b = \int f d\mathcal{U}$ , which gives

$$W = f^{-1}(b) = f^{-1}(\int f d\mathcal{U}) \in \mathcal{U}$$

so  $\mathcal{U} = \mathcal{V}$ , that is,  $\mathcal{U}$  is maximal.  $\square$

To prove the converse of the above lemma we will rely on Proposition 2.4.6 which uses the assumption that our theory contains the theory of groups to prove that images of affine cofinite sets under surjections are affine cofinite.

**Lemma 3.3.2.** *A maximal proper filter on an algebra  $A$  is an ultrafilter on  $A$ .*



*Proof.* Let  $\mathcal{V}$  be a maximal proper filter on the algebra  $A$ . Suppose also that we have a morphism  $A \xrightarrow{f} B$  with  $B$  finite. First we will show that there is a  $b \in B$  such that  $f^{-1}(b)$  has the finite intersection property with all the elements of  $\mathcal{V}$ .

We can factorise  $f$  as

$$\begin{array}{ccc} & B' & \\ f' \nearrow & & \searrow i \\ A & \xrightarrow{f} & B \end{array}$$

where  $f'$  is a surjection and  $i$  is an injection. As subalgebras of finite algebras are also finite  $B'$  will be finite. If we can show that there is a  $b \in B'$  such that  $f'^{-1}(b)$  has the finite intersection property with all the elements of  $\mathcal{V}$  then  $f^{-1}(i(b))$  will also have this property as we have  $f^{-1}(i(b)) = f'^{-1}(i^{-1}(i(b))) = f'^{-1}(b)$ . So we may assume that  $f$  is surjective.

Consider the set

$$f(\mathcal{V}) = \{f(S) | S \in \mathcal{V}\}$$

by Proposition 2.4.6 this is a family of affine cofinite subsets of  $B$ . However,  $B$  is finite, so by Proposition 2.3.5, there is some finite subset  $\mathcal{W} \subset \mathcal{V}$  such that

$$\bigcap_{S \in \mathcal{V}} f(S) = \bigcap_{S \in \mathcal{W}} f(S).$$

As  $\mathcal{V}$  is a filter and  $\mathcal{W}$  contains finitely many elements we have

$$\bigcap_{S \in \mathcal{W}} S \neq \emptyset.$$

However we also have

$$f\left(\bigcap_{S \in \mathcal{W}} S\right) \subset \bigcap_{S \in \mathcal{W}} f(S)$$

so, as the left hand side of this inclusion is non-empty, we have

$$\bigcap_{S \in \mathcal{V}} f(S) = \bigcap_{S \in \mathcal{W}} f(S) \neq \emptyset.$$

Choose a point  $b \in B$  in the above intersection. So, we have  $b \in f(S)$  for all  $S \in \mathcal{V}$ , that is, for all  $S \in \mathcal{V}$ , there is some  $x \in S$  with  $f(x) = b$ . This is equivalent to

$$f^{-1}(b) \cap S \neq \emptyset.$$

So the family of sets  $\{f^{-1}(b)\} \cup \mathcal{V}$  has the finite intersection property and can be extended to a proper filter  $\mathcal{V}'$  of affine cofinite sets. However, as  $\mathcal{V}$  is maximal, we must have  $\mathcal{V} = \mathcal{V}'$ , so  $f^{-1}(b) \in \mathcal{V}$ .

We have shown that for any morphism  $A \xrightarrow{f} B$  there exists a  $b \in B$  with  $f^{-1}(b) \in \mathcal{V}$ . To see that such a point is necessarily unique, suppose we have elements  $b$  and  $b'$  of  $B$ , with  $f^{-1}(b)$  and  $f^{-1}(b')$  in  $\mathcal{V}$  then, as  $\mathcal{V}$  is proper, we must have  $f^{-1}(b) \cap f^{-1}(b') \neq \emptyset$  so  $b = b'$ .

To prove that  $\mathcal{V}$  satisfies the final condition of the definition of an ultrafilter suppose we have an affine cofinite subset  $S \subset B$  with  $f^{-1}(S) \in \mathcal{V}$ . Let  $B \xrightarrow{h} C$  be a morphism with  $C$  finite such that  $S = h^{-1}(c)$  for some  $c \in C$ . We have  $f^{-1}(h^{-1}(c)) \in \mathcal{V}$  and  $f^{-1}(b) \in \mathcal{V}$  so, as  $\mathcal{V}$  is

proper, we have

$$f^{-1}(h^{-1}(c)) \cap f^{-1}(b) \neq \emptyset$$

so  $b \in h^{-1}(c) = S$ , so  $\mathcal{V}$  is an ultrafilter. □

Combining the above two results we have the following.

**Proposition 3.3.3.** *A family of affine cofinite subsets of an algebra is an ultrafilter if and only if it is a maximal proper filter.*

*Proof.* This is a direct consequence of Lemmas 3.3.1 and 3.3.2. □

So far we have only been able to construct principal ultrafilters on algebras. However we can use the same arguments as in the set case to show that there exists an ultrafilter containing any family of affine cofinite sets with the finite intersection property.

**Proposition 3.3.4.** *Given a family of affine cofinite subsets of some algebra with the finite intersection property, we can extend them to an ultrafilter on that algebra.*

*Proof.* Given an indexing set  $\lambda$  and a family  $(T_i)_{i \in \lambda}$  of affine cofinite subset of  $A$ , we can extend them to a proper filter of affine cofinite subsets of  $A$  as we do in Lemma 3.1.7 for the set case. By the same argument as used in Lemma 3.1.8 we know that this filter is contained in a proper maximal filter of affine cofinite sets, which we denote  $\mathcal{U}$ . However by Lemma 3.3.2 this  $\mathcal{U}$  will be an ultrafilter on  $A$ . □

## Chapter 4

# Algebraically Compact Algebras

We want to generalise the notions of compact Hausdorff space and linearly compact vector space. To do this we will define algebraically compact algebras. These are topological algebras that satisfy certain “compactness” conditions.

Compact Hausdorff spaces and ultrafilters are closely related. The compact Hausdorff spaces are exactly the spaces on which every ultrafilter has a unique limit. We have already defined the notion of an ultrafilter on an algebra and in this chapter we will define the notion of a limit of an ultrafilter on an algebra. We will see that, on algebraically compact spaces, ultrafilters have a unique limit.

### 4.1 Compact Hausdorff Spaces

First we consider ultrafilters on sets and how they relate to compact Hausdorff spaces.

**Definition 4.1.1.** *Given an ultrafilter  $\mathcal{U}$  on a topological space  $S$  a point  $s$  of  $S$  is a **limit** of  $\mathcal{U}$  if and only if every open subset of  $S$  containing  $s$  is in  $\mathcal{U}$ .*

In the following lemma we see that there is an equivalent characterisation of the limit of an ultrafilter involving closed sets.

**Lemma 4.1.2.** *A limit of an ultrafilter  $\mathcal{U}$  is a point contained in the intersection of the closed sets of  $\mathcal{U}$ .*

*Proof.* Suppose that we have point  $p$  such that, for all open sets  $W$ , if  $p$  is in  $W$  then  $W$  is an element of  $\mathcal{U}$ , that is,

$$p \in W \Rightarrow W \in \mathcal{U}.$$

Taking the contrapositive, this is equivalent to

$$W \notin \mathcal{U} \Rightarrow p \notin W.$$

As  $\mathcal{U}$  is an ultrafilter,  $W \notin \mathcal{U}$  is equivalent to  $W^c \in \mathcal{U}$ , so we can rewrite the above as

$$W^c \in \mathcal{U} \Rightarrow p \in W^c.$$

So, as the closed sets are exactly the complements of the open sets, we can equivalently say that for every closed set  $V$ , if  $V$  is in  $\mathcal{U}$  then  $p$  is in  $V$ . We then have that, for some point  $p$ , every open set containing  $p$  is in  $\mathcal{U}$  if and only if  $p$  is in the intersection of the closed sets of  $\mathcal{U}$ . That is,  $p$  is a limit of  $\mathcal{U}$  if and only if it is in the intersection of the closed sets of  $\mathcal{U}$ .  $\square$

We will frequently need to refer to the set of open or closed sets in an ultrafilter, so we introduce the following notation.

**Definition 4.1.3.** For an ultrafilter  $\mathcal{U}$  we write  $\mathcal{O}(\mathcal{U})$  for the set of open sets in  $\mathcal{U}$  and  $\mathcal{C}(\mathcal{U})$  for the set of closed sets in  $\mathcal{U}$ .

The above notation applies both to ultrafilters on sets and, more generally, ultrafilters on algebras.

An arbitrary ultrafilter on a topological space may not have a limit. If such an arbitrary ultrafilter does have a limit it may not be unique. However, for a certain class of spaces, compact Hausdorff spaces, the limit of an ultrafilter always exist and is always unique.

**Lemma 4.1.4.** On a compact Hausdorff space every ultrafilter has a unique limit.

*Proof.* Suppose  $\mathcal{U}$  is an ultrafilter on a compact Hausdorff space  $A$ . As  $\mathcal{U}$  is a filter the family of sets  $\mathcal{C}(\mathcal{U})$  will have the finite intersection property, as these sets are closed and  $A$  is compact the intersection

$$L = \bigcap_{S \in \mathcal{C}(\mathcal{U})} S$$

is non-empty. So, by Lemma 4.1.2,  $\mathcal{U}$  has at least one limit.

Suppose we have two points,  $x$  and  $y$  in  $L$ . By Lemma 4.1.2, these points are both limits of  $\mathcal{U}$  so every open set containing  $x$  or  $y$  will be in  $\mathcal{U}$ . However, as  $A$  is Hausdorff there are open sets  $U_x$  and  $U_y$  with  $x \in U_x$  and  $y \in U_y$  such that  $U_x \cap U_y = \emptyset$ , by the above we will have

$$\emptyset = U_x \cap U_y \in \mathcal{U},$$

contradicting the properness of  $\mathcal{U}$ . So the intersection of the closed sets of  $\mathcal{U}$  contains a single element, that is,  $\mathcal{U}$  has a unique limit.  $\square$

When considering ultrafilters on compact Hausdorff spaces we denote the limit of an ultrafilter  $\mathcal{U}$  by  $\lim(\mathcal{U})$ . We now prove the converse of the above result.

**Lemma 4.1.5.** If every ultrafilter on a topological space  $A$  has a unique limit then  $A$  is compact Hausdorff.

*Proof.* Suppose we have a family of closed subsets  $(S_i)_{i \in \lambda}$  of  $A$  with the finite intersection property. We can extend  $(S_i)_{i \in \lambda}$  to an ultrafilter  $\mathcal{U}$ . This ultrafilter will have a limit,  $\lim(\mathcal{U})$ , by Lemma 4.1.2 we have

$$\lim \mathcal{U} \in \bigcap_{S \in \mathcal{C}(\mathcal{U})} S \subset \bigcap_{i \in \lambda} S_i$$

so the intersection of the sets  $(S_i)_{i \in \lambda}$  is non-empty, so  $A$  is compact.

Now suppose we have two points  $x$  and  $y$  in  $A$  where the Hausdorffness condition fails. That is, there are no pair of open sets  $U_x$  and  $U_y$  with  $x \in U_x$  and  $y \in U_y$  such that  $U_x \cap U_y = \emptyset$ . This

means that the family of open sets that contain either  $x$  or  $y$  will have the finite intersection property so we can extend them to an ultrafilter  $\mathcal{V}$ .

Every open set that contains  $x$  is in  $\mathcal{V}$ , so  $x$  is the limit of  $\mathcal{V}$ , but we also have that every open set that contains  $y$  is in  $\mathcal{V}$ , so  $y$  is the limit of  $\mathcal{V}$ . As limits of ultrafilters on this space are unique we have  $x = y$  and so, our space is Hausdorff.  $\square$

Combining the above lemmas we have the following result.

**Proposition 4.1.6.** *The compact Hausdorff spaces are exactly the topological spaces on which every ultrafilter has a unique limit.*

*Proof.* This follows directly from Lemmas 4.1.4 and 4.1.5.  $\square$

We have not defined algebraically compact algebras yet, but we will show that on an algebraically compact algebra every ultrafilter has a unique limit.

**Lemma 4.1.7.** *A finite topological space is compact Hausdorff if and only if it is discrete.*

*Proof.* Suppose we have a finite discrete topological space  $S$ . The discrete topology is Hausdorff and any topology on a finite set is compact. So  $S$  is compact Hausdorff.

In the other direction, suppose we have a finite compact Hausdorff space  $T$ . Suppose also that we have a point  $t$  in  $T$ . For any point  $s$  in  $T$  with  $s \neq t$ , by Hausdorffness there is an open set  $U_s$  containing  $t$  with  $s \notin U_s$ . However, as  $T$  is finite, there are only finitely many such sets. So the set

$$\bigcap_{\substack{s \in T \\ s \neq t}} U_s = \{t\}$$

is open. So  $T$  is discrete.  $\square$

## 4.2 Linearly Compact Vector Spaces

Linearly compact vector spaces were introduced by Lefschetz in [12], a good reference is [7]. In the theory of algebraically compact algebras, linearly compact vector spaces are an important special case. We cover them in detail below.

By way of motivation, let  $V$  be a vector space over a field  $k$ . Consider the dual space,  $V^* = \text{Vect}(V, k)$ , of  $V$ , this will also have the structure of a vector space. If  $V$  is finite dimensional we have a canonical natural isomorphism

$$V \cong (V^*)^*.$$

Suppose now that  $V$  is a vector space over the field  $k$  of infinite dimension  $\lambda$ . Then we can think of elements of  $V$  as  $\lambda$  length sequences of elements of  $k$  with only finitely many non-zero entries. That is, we have

$$V \cong \bigoplus_{i \in \lambda} k.$$

Now consider the dual vector space  $V^*$ . Given a basis for  $V$ , a function from  $V$  to  $k$  is uniquely defined by its values at the basis. So an element of  $V^*$  is a  $\lambda$  length sequence of

elements of  $k$ . So we have

$$V^* \cong \prod_{i \in \lambda} k.$$

However, as  $\lambda$  is infinite we then have that the dimension of the dual space,  $V^*$ , is larger than the dimension of  $V$ , so we have

$$V \not\cong (V^*)^*.$$

However, there is a way of defining the dual of a vector space such that every vector space (be it of finite or infinite dimension) is isomorphic to its double dual. Suppose now that the vector space  $V$  is a topological vector space and that the field  $k$  has the discrete topology.

**Definition 4.2.1.** *Given a topological vector space  $V$ , we define  $V^\dagger$  as the topological dual of  $V$ , that is, it is the vector space of continuous linear maps from  $V$  to  $k$ .*

We let the annihilators of the finite dimensional subspaces of  $V$  to be a basis for a topology on  $V^*$ . When  $V^*$  is equipped with this topology we have the following isomorphism

$$V \cong (V^*)^\dagger.$$

For the details of the above construction see [7], in particular Proposition 24.4. We can characterise these vector spaces  $V^*$  with the topology described above, they are the linearly compact vector spaces.

**Definition 4.2.2.** *A topological vector space is **linearly compact** if it satisfies the following properties*

- *It has a basis of open affine sets*
- *It is Hausdorff*
- *Any family of affine sets with the finite intersection property has non-empty intersection.*

What can we say about the topology of a linearly compact vector space? Well, suppose we have an open affine set  $U$  in a linearly compact vector space  $W$ . Then, as vector addition is continuous, for all  $v$  in  $W$  we have that  $v + U$  is open, so all the cosets of  $U$  will also be open.

However we have

$$U = \left( \bigcup_{v \in W \setminus U} v + U \right)^c$$

so  $U$  will also be closed. So, in a linearly compact vector space every open affine set is closed.

Suppose now that we have a closed subspace  $T$  of a linearly compact vector space  $W$ . Then for some indexing set  $\lambda$  and some family of open affine sets  $(S_i)_{i \in \lambda}$  we have

$$T = \left( \bigcup_{i \in \lambda} S_i \right)^c = \bigcap_{i \in \lambda} S_i^c.$$

And as  $0 \in T$  we have  $0 \in S_i^c$  for all  $i \in \lambda$ . For any  $i \in \lambda$ , as  $S_i^c$  is the union of the non-trivial cosets of  $S_i$ , it is also open. So the closed subspaces of a linearly compact vector space are arbitrary intersections of open subspaces.

We have seen that a vector space  $V$  is finite dimensional if and only if we have  $V \cong (V^*)^*$ . In the following two lemmas we will see how we can also characterise the finite dimensional vector spaces in terms of affine sets.

**Lemma 4.2.3.** *A vector space is finite dimensional if and only if every descending chain of affine sets stabilises.*

*Proof.* Suppose we have a finite dimensional vector space  $V$  and a descending chain of affine subspaces  $(S_i)_{i \in \mathbb{N}}$  of  $V$ . For all  $i \in \mathbb{N}$  we have  $\dim(S_i) \leq \dim(V)$ . However, for all  $i \in \mathbb{N}$  we also have  $\dim(S_{i+1}) \leq \dim(S_i)$ . So there is some finite  $n$  such that the dimension of  $S_i$  for  $i \geq n$  is constant. However this gives that the  $S_i = S_n$  for all  $i \geq n$ .

Suppose we have an infinite dimensional vector space  $V$ . Suppose we have a basis for  $V$ . Let  $S_i$  be the set of vectors that are zero in the first  $i$  coordinates. These will be an infinite descending chain that does not stabilise.  $\square$

In particular, as a consequence of the above lemma, we have that the finite dimensional vector spaces satisfy the conditions of Definition 2.3.3, that is they are a subcategory of finite algebras. In the following lemma, we give an interesting condition that is equivalent to the affine descending chain condition.

**Lemma 4.2.4.** *A vector space is finite dimensional if and only if any family of affine sets with the finite intersection property has non-empty intersection.*

*Proof.* Suppose  $V$  is a finite dimensional vector space and for some indexing set  $\lambda$  we have a family of affine sets  $(S_i)_{i \in \lambda}$  with the finite intersection property. Consider the family of affine sets defined by

$$T_i = \bigcap_{0 \leq j \leq i} S_j.$$

This is a descending chain so, by Lemma 4.2.3, there is a finite  $n$  such that for all  $i \geq n$  we have  $T_i = T_n$ . However this gives

$$\bigcap_{i \in \lambda} S_i = \bigcap_{i \in \lambda} T_i = T_n \neq \emptyset.$$

On the other hand, suppose that  $V$  is a vector space of infinite dimension  $\lambda$ . Suppose we have some basis for  $V$ . Let  $S_i$  be the affine subset of  $V$  consisting of all vectors with a 1 in the  $i^{\text{th}}$  coordinate. These sets will have the finite intersection property but we have

$$\bigcap_{i \in \lambda} S_i = \emptyset.$$

$\square$

As we will see in the following Lemma, which is adapted from Proposition 20 of [17], in a linearly compact vector space, we can restrict our attention to open affine cofinite sets.

**Lemma 4.2.5.** *In a linearly compact vector space the open affine sets are finite codimensional.*

*Proof.* Suppose that  $U$  is an open affine subset of a linearly compact vector space  $W$ . As  $U$  is affine it is a coset of some open subspace  $U'$  of  $W$ . Consider the quotient vector space  $W/U'$ . Let  $W \xrightarrow{p} W/U'$  be the canonical projection morphism.

Suppose for some indexing set  $\lambda$  we have a family  $(S_i)_{i \in \lambda}$  of affine subsets of  $W/U'$  with the finite intersection property. The sets  $p^{-1}(S_i)$  will also have the finite intersection property. However, as  $W/U$  is discrete and the sets  $p^{-1}(S_i)$  will be closed, so by the definition of linear compactness we have

$$\bigcap_{i \in \lambda} p^{-1}(S_i) \neq \emptyset.$$

However, this gives

$$\bigcap_{i \in \lambda} S_i \neq \emptyset.$$

So by Lemma 4.2.4  $W/U'$  is finite, so  $U$  is cofinite.  $\square$

As we will now see, there are in fact three different equivalent ways to characterise the linearly compact vector spaces. First however, we make the following definitions.

**Definition 4.2.6.** Suppose  $W$  is a vector space, then  $\mathcal{N}_W$  is the category with objects the open subspaces of  $W$ , and morphisms the inclusions of such subspaces.

**Definition 4.2.7.** Suppose  $W$  is a vector space, then  $J_W$  is the functor from  $\mathcal{N}_W$  to  $\mathbf{Vect}$  that sends an object  $U$  of  $\mathcal{N}_W$  to the quotient space  $W/U$ , and sends the inclusions of these subspaces to the corresponding canonical morphisms between quotient spaces.

The following Lemma is adapted from Proposition 24.4 of [7].

**Lemma 4.2.8.** For a topological vector space  $W$  the following are equivalent

- $W$  is linearly compact
- $W$  is a limit of finite dimensional discrete vector spaces
- $W$  is a dual space with the function topology.

*Proof.* Suppose that  $W$  is a linearly compact vector space. Given an object  $U$  of  $\mathcal{N}_W$  the quotient spaces  $W/U$  will be discrete and, by Lemma 4.2.5, finite dimensional, so  $\lim J_W$  is a limit of finite dimensional discrete vector spaces. We will show that  $W$  is isomorphic to this limit.

There is a cone from  $W$  to the diagram  $\mathcal{N}_W$ , composed of the canonical morphisms  $W \rightarrow W/U$  for  $U$  an object of  $\mathcal{N}_W$ . This cone induces a morphism  $W \xrightarrow{e} \lim J_W$ , we will show that this morphism is an isomorphism.

First suppose we have two elements  $w$  and  $w'$  of  $W$ , with  $w \neq w'$ . As  $w - w' \neq 0$ , by Hausdorffness there is some open linear subspace  $U$  of  $W$  with  $w - w' \notin U$ . So we have  $w + U \neq w' + U$ . This gives that  $e(w) \neq e(w')$ . So  $e$  is injective.

To see that  $e$  is surjective suppose we have a vector  $v$  in  $\lim J_W$ . That is, for each object  $V$  of  $\mathcal{N}_W$  we have an element  $v_V + V$  of  $W/V$ , such that if we have objects  $V$  and  $V'$  of  $\mathcal{N}_W$  with  $V \subset V'$  then  $v_V + V = v_{V'} + V'$ .

Suppose  $U$  and  $U'$  are open affine subspaces of  $W$ , then  $U \cap U'$  is also an open affine subspace of  $W$  and  $U \cap U' \subset U$  and  $U \cap U' \subset U'$ . So we have

$$v_{U \cap U'} + U = v_U + U$$



and

$$v_{U \cap U'} + U' = v_{U'} + U'.$$

This gives

$$(v_U + U) \cap (v_{U'} + U') = (v_{U \cap U'} + U) \cap (v_{U \cap U'} + U') \neq \emptyset.$$

So the affine sets  $v_U + U$ , for  $U$  an open affine subset of  $W$ , have the finite intersection property. They are also closed, so, as  $W$  is linearly compact, this means that they have non-empty intersection. Let  $l$  be a point in this intersection. So for all  $U$  of  $\mathcal{N}_W$  we have  $l \in v_U + U$ , which gives,

$$l + U = v_U + U.$$

However, by the definition of  $e$  we have

$$e(l)_U + U = l + U.$$

So for all  $U$  in  $\mathcal{N}_W$  we have

$$e(l)_U + U = v_U + U$$

that is,  $e(l) = v$ . So  $e$  is surjective. Now, by construction  $W$  and  $\lim J_W$  have the same topology and so we have  $W \cong \lim J_W$ .

Suppose that  $W$  is a limit of finite dimensional discrete vector spaces. That is  $W = \lim F$ , for some diagram  $F$  of finite dimensional discrete spaces. Let  $G$  be the diagram consisting of the dual of the objects and morphisms of  $F$ , then we have

$$W = \lim F = \text{Vect}(\text{colim} G, k).$$

So  $W$  is a dual space with the function topology.

Suppose now that we have some vector space  $V$ , we will show that  $V^*$  with the function topology is linearly compact. For  $v \in V$ ,  $\text{ev}_v$  is a linear map from  $V^*$  to  $k$ . Suppose we have a finite dimensional subspace  $S$  of  $V$ . Suppose also that for some finite indexing set  $I$ ,  $(s_i)_{i \in I}$  is a basis for  $S$ , then we have

$$\text{Ann}(S) = \bigcap_{i \in I} \text{ev}_{s_i}^{-1}(0).$$

So we can take the linear subspaces  $\text{ev}_v^{-1}(0)$  for  $v \in V$  as a basis for the topology on  $V^*$ .

We have already seen that the closed subspaces of  $V^*$  are arbitrary intersections of open subspaces. However in  $V^*$  the open subspaces are the annihilators of finite dimensional subspaces of  $V$ , and given finite dimensional subspaces  $A$  and  $B$  of  $V$  we have

$$\text{Ann}(A) \cap \text{Ann}(B) = \text{Ann}(A \cup B).$$

So the closed subspaces of  $V^*$  are exactly the annihilators of arbitrary subspaces of  $V$ .

Now suppose for some indexing set  $\lambda$  we have a family  $(T_i)_{i \in \lambda}$  of closed affine subsets of  $V^*$  with the finite intersection property. The sets  $(T_i)_{i \in \lambda}$  are the cosets of the annihilators of arbitrary subspaces of  $V$ . So for  $i \in \lambda$  we have an element  $f_i \in V^*$  and a subspace  $S_i \subset V$  such that  $T_i = f_i + \text{Ann}(S_i)$ .

Given  $i$  and  $j$  in  $\lambda$  we have

$$(f_i + \text{Ann}(S_i)) \cap (f_j + \text{Ann}(S_j)) = T_i \cap T_j \neq \emptyset.$$

Suppose we have some  $h$  in this intersection. Then for  $v \in S_i \cap S_j$  we have  $h(v) = f_i(v)$  and  $h(v) = f_j(v)$ , which gives  $f_i(v) = f_j(v)$ . So the  $(f_i)_{i \in \lambda}$  agree on the intersections of the  $S_i$ .

We let  $f$  be the map from  $U_{i \in \lambda} S_i$  to  $k$  that agrees with  $f_i$  on  $S_i$ . We can extend this map to an element  $f$  of  $V^*$ . We then have

$$f \in \bigcap_{i \in \lambda} (f_i + \text{Ann}(S_i)) = \bigcap_{i \in \lambda} T_i$$

so the intersection of the  $(T_i)_{i \in \lambda}$  is non-empty.

Suppose we have vectors  $f$  and  $g$  in  $V^*$ , with  $f \neq g$ . Then there must be some  $v \in V$  with  $f(v) \neq g(v)$ . We have that  $v$  generates a finite dimensional subspace  $S$  so consider the open subspace  $\text{Ann}(S)$  of  $V^*$ . We have  $f \in f + \text{Ann}(S)$  and  $g \in g + \text{Ann}(S)$  and these are open subsets of  $V^*$ . Suppose we have some element

$$h \in f + \text{Ann}(S) \cap g + \text{Ann}(S)$$

then we would have

$$f(v) = h(v) = g(v)$$

a contradiction. So we have

$$(f + \text{Ann}(S)) \cap (g + \text{Ann}(S)) = \emptyset.$$

So  $V^*$  is Hausdorff. Putting this all together, we have that  $V^*$  is linearly compact. □

We saw in Lemma 4.1.7 that there is exactly one compact Hausdorff topology on a finite set, the discrete topology. Similarly, we now see that there is exactly one linearly compact topology on a finite dimensional vector space, the discrete topology. This lemma is adapted from Proposition 19 of [17].

**Lemma 4.2.9.** *A finite dimensional vector space is linearly compact if and only if it is discrete.*

*Proof.* Suppose  $W$  is a discrete finite dimensional vector space. The one point sets will be a basis for this topology. These sets are affine so our topology has an affine basis. The discrete topology is certainly Hausdorff. Now, by Lemma 4.2.4, any family of affine subsets with the finite intersection property has non-empty intersection, so  $W$  is linearly compact.

Suppose now we have a linearly compact finite dimensional vector space  $W$ . For some indexing set  $\lambda$  consider the family  $(S_i)_{i \in \lambda}$  of open affine sets containing  $\{0\}$ . Given a non-zero vector  $v$  in  $V$ , by Hausdorffness there is some open affine set  $U_v$  with  $0 \in U_v$  and  $v \notin U_v$ . So we have

$$\bigcap_{i \in \lambda} S_i = \{0\}.$$

Consider the affine open sets

$$T_i = \bigcap_{0 \leq j \leq i} S_j$$

these will give a descending chain so by 4.2.3 there is some finite  $n$  such that for all  $i \geq n$  we have  $T_i = T_n$ . So we have

$$\bigcap_{0 \leq i \leq n} S_i = \bigcap_{i \in \lambda} S_i = \{0\}.$$

So, as it is the intersection of finitely many open affine sets we have that  $\{0\}$  is open, so the topology is discrete.  $\square$

### 4.3 Defining Algebraically Compact Algebras

We now define our topological algebras that generalise compact Hausdorff spaces and linearly compact vector spaces.

**Definition 4.3.1.** *A topological algebra  $A$  is **algebraically compact** if it satisfies the following conditions*

1. *The open affine cofinite subsets are a basis for the topology*
2. *The topology is Hausdorff*
3. *Any family of closed affine cofinite subsets with the finite intersection property has non-empty intersection.*

We will prove below that the restriction in the third condition of algebraic compactness to affine subsets that are cofinite is redundant.

In this thesis, when we are not considering some specific algebraic theory such as the theory of sets or the theory of vector spaces, we assume only that we have some algebraic theory containing the theory of groups. When this is the case, we let **AlgComp** be the category of algebraically compact algebras for this theory and continuous morphisms between them.

Let us suppose that  $\mathcal{A}$  is the category **Set** of sets and functions and that  $\mathcal{B}$  is the subcategory of finite sets. Any subset of a set can be written as a fibre of a morphism into the two element set which is finite, so every subset is affine cofinite. From this we see that the first condition of algebraic compactness is trivially satisfied by any topological space and the third condition reduces to ordinary compactness. So an algebraically compact set is a compact Hausdorff space.

In Lemma 4.1.7 we saw that there is exactly one compact Hausdorff topology on a finite set, the discrete topology. Similarly, in Lemma 4.2.9 we saw that there is exactly one linearly compact topology on a finite dimensional vector space, the discrete topology. We now prove an analogous result in the general case that we will rely on later.

**Proposition 4.3.2.** *A finite algebra is algebraically compact if and only if it is discrete.*

*Proof.* Suppose that  $B$  is a finite algebra with the discrete topology. The one point sets are a basis for the discrete topology, as these are fibres of the identity morphism, so the discrete topology has an affine cofinite basis. The discrete topology is always Hausdorff so we just have to prove that it satisfies the third condition of algebraic compactness.

Suppose we have an indexing set  $\lambda$  and a family of closed affine cofinite subsets  $(S_i)_{i \in \lambda}$  of  $B$  with the finite intersection property. As  $B$  is finite, by Proposition 2.3.5 there is a finite subset  $I \subset \lambda$  with

$$\bigcap_{i \in \lambda} S_i = \bigcap_{i \in I} S_i,$$

however the subsets  $(S_i)_{i \in \lambda}$  satisfy the finite intersection property so we have

$$\bigcap_{i \in \lambda} S_i \neq \emptyset.$$

That is,  $B$  is algebraically compact.

In the other direction, suppose that  $B$  is a finite algebraically compact algebra. Suppose we have a point  $b$  in  $B$ . By Hausdorffness, for all points  $c$  in  $B$  with  $c \neq b$ , there is an open affine cofinite set  $U_c$  with  $b \in U_c$  and  $c \notin U_c$ . Consider the intersection

$$\bigcap_{\substack{c \in B \\ c \neq b}} U_c = \{b\}.$$

As  $B$  is finite, by Proposition 2.3.5, there is some finite set  $S$  of elements of  $B$  with

$$\bigcap_{\substack{c \in S \\ c \neq b}} U_c = \bigcap_{\substack{c \in B \\ c \neq b}} U_c = \{b\}.$$

However this is an intersection of finitely many open sets, so  $\{b\}$  is an open set, so  $B$  is discrete.  $\square$

We will prove that in an algebraically compact algebra an affine cofinite set is open if and only if it is closed. To do this we repeatedly rely on the assumption that our theory contains the theory of groups. In the following lemma we prove one direction of the equivalence.

**Lemma 4.3.3.** *The open affine cofinite subsets of  $A$  are closed.*

*Proof.* If  $S$  is an open affine subset of  $A$  then, as the group operation is continuous, all the cosets of  $S$  are also open. We let  $R$  be the union of all the non-trivial cosets of  $S$ , this will be open. However we can write  $S$  as

$$S = A \setminus R$$

so  $S$  is closed.  $\square$

To prove the converse of this result, that the closed affine cofinite sets are open, we will require the following four lemmas.

**Lemma 4.3.4.** *Suppose we have algebras  $A$  and  $B$ , a morphism  $A \xrightarrow{f} B$  and a subset  $U$  of  $A$  then we have*

$$f^{-1}f(U) = \bigcup_{s \in \ker f} s \cdot U.$$

*Proof.* Given a point  $x$  in  $f^{-1}f(U)$  we have  $f(x) = f(u)$  for some  $u \in U$ . This gives  $f(x \cdot u^{-1}) = 1_B$ , that is  $x \cdot u^{-1} \in \ker f$ . We also have

$$x = x \cdot u^{-1} \cdot u \in x \cdot u^{-1} \cdot U$$

putting these facts together gives

$$x \in \bigcup_{s \in \ker f} s \cdot U.$$

In the other direction suppose we have an  $x$  with

$$x \in \bigcup_{s \in \ker f} s \cdot U.$$

Then we have  $x = s \cdot u$  for some  $s$  in  $\ker f$  and  $u$  in  $U$  which gives

$$f(x) = f(s \cdot u) = f(s) \cdot f(u) = f(u).$$

So we have  $x \in f^{-1}(f(U))$ . □

**Lemma 4.3.5.** *If  $A \xrightarrow{f} B$  is a surjective morphism of topological algebras, and  $B$  has the quotient topology, then  $f$  is open.*

*Proof.* Suppose we have an open subset  $U$  of  $A$ . By Lemma 4.3.4 we have that

$$f^{-1}f(U) = \bigcup_{s \in \ker f} s \cdot U.$$

However,  $U$  is open, so  $s \cdot U$  is also open which means that  $f^{-1}f(U)$  is open. As  $f$  is a quotient morphism we then have that  $f(U)$  is open, so  $f$  is an open morphism. □

**Lemma 4.3.6.** *Suppose we have a topological algebra in which every singleton is closed, then the topology is Hausdorff.*

*Proof.* Suppose we have an algebra  $A$  in which all points are closed. Consider the continuous morphism  $f$  that sends a pair of points  $x$  and  $y$ , to  $x \cdot y^{-1}$ . We have that the point  $\{1_A\}$  is closed so  $f^{-1}(1_A)$  is closed in  $A \times A$ . That is the set  $\Delta = \{\langle x, x \rangle | x \in A\}$  is closed in  $A$ .

Now suppose we have points,  $a$  and  $b$  in  $A$  with  $a \neq b$ . Then, by the above, there is an open set  $U$  of  $A \times A$  with  $\langle a, b \rangle \in U$  and  $U \not\subset \Delta$ . So, as  $A \times A$  has the product topology, there are open sets  $U_1$  and  $U_2$  with  $a \in U_1$  and  $b \in U_2$  such that  $U_1 \cap U_2 = \emptyset$ . That is,  $A$  is Hausdorff. □

The quotient of a compact Hausdorff space is also compact Hausdorff, the following is a similar result for algebraically compact algebras.

**Lemma 4.3.7.** *If  $A$  is an algebraically compact algebra and we have a surjective morphism  $A \xrightarrow{f} B$  with  $B$  finite and an element  $b \in B$  such that  $f^{-1}(b)$  is closed, then the quotient topology on  $B$  is algebraically compact.*

*Proof.* We will show that  $B$  satisfies each of the conditions of algebraic compactness in turn. First suppose we have an open set  $U$  in  $B$ . By the definition of the quotient topology we have that  $f^{-1}(U)$  is open in  $A$ . As  $A$  is algebraically compact there is an indexing set  $\lambda$  and a family of open affine cofinite sets  $(V_i)_{i \in \lambda}$  with

$$f^{-1}(U) = \bigcup_{i \in \lambda} V_i.$$

However as  $f$  is surjective we have that  $U = f f^{-1}(U)$  so we have

$$U = f f^{-1}(U) = f\left(\bigcup_{i \in \lambda} V_i\right) = \bigcup_{i \in \lambda} f(V_i).$$

Now by Lemma 4.3.5 and Proposition 2.4.6 the sets  $f(V_i)$  are open affine cofinite. So, if for some indexing set  $\zeta$  the sets  $(V_i)_{i \in \zeta}$  are a basis of open affine cofinite sets for  $A$  then the sets  $(f(V_i))_{i \in \zeta}$  will be a basis of open affine cofinite sets for  $B$ .

Suppose for some indexing set  $\gamma$  we have a family  $(S_i)_{i \in \gamma}$  of closed affine cofinite subsets of  $B$  with the finite intersection property. As  $f$  is surjective, for any finite subset  $I$  of  $\gamma$  we have

$$\bigcap_{i \in I} f^{-1}(S_i) = f^{-1}\left(\bigcap_{i \in I} S_i\right) \neq \emptyset.$$

So, as  $f$  is continuous, we have a family of closed affine cofinite sets  $(f^{-1}(S_i))_{i \in I}$  with the finite intersection property in  $A$ . However,  $A$  is algebraically compact, so we have

$$\bigcap_{i \in \gamma} f^{-1}(S_i) \neq \emptyset.$$

Given some point  $x$  in the above intersection we have  $f(x) \in S_i$  for all  $i \in \gamma$ , so we have

$$\bigcap_{i \in \gamma} S_i \neq \emptyset,$$

so  $B$  satisfies the third condition of algebraic compactness.

Now, there is some  $b$  in  $B$  with  $f^{-1}(b)$  closed in  $A$ . As  $B$  has the quotient topology this means that the set  $\{b\}$  is closed in  $B$ . By Lemma 4.3.6  $B$  is Hausdorff. So we have seen that  $B$  is algebraically compact.  $\square$

Using the previous lemma we can now prove the converse of Lemma 4.3.3.

**Lemma 4.3.8.** *In an algebraically compact algebra if an affine cofinite set is closed it is open.*

*Proof.* Suppose we have an algebraically compact algebra  $A$  and a closed affine cofinite subset  $U$  of  $A$ . We can write  $U$  as  $f^{-1}(b)$  for some surjective morphism  $A \xrightarrow{f} B$  with  $B$  finite. If we endow  $B$  with the quotient topology we see by Lemma 4.3.7 that  $B$  is algebraically compact. However, by Proposition 4.3.2, the only algebraically compact topology on a finite algebra is discrete, so the set  $\{b\}$  is open, which means that  $U = f^{-1}(b)$  is also open.  $\square$

So we now have the following result.

**Proposition 4.3.9.** *In an algebraically compact algebra, an affine cofinite set is open if and only if it is closed.*

*Proof.* This follows from Lemmas 4.3.3 and 4.3.8.  $\square$

In the definition of linearly compact vector space there is no reference to ‘cofiniteness’. We will now see that we can extend the third condition of algebraic compactness to apply to all affine sets, not just the affine cofinite ones.

**Lemma 4.3.10.** *In an algebraically compact algebra a closed affine set can be written as the intersection of closed affine cofinite sets.*

*Proof.* Suppose we have a closed affine set  $V$ . As  $V$  is an affine set there is a surjective morphism  $A \xrightarrow{f} B$  and a point  $b$  in  $B$  such that  $V = f^{-1}(b)$ . From Lemma 4.3.7 we see that the quotient

topology on  $B$  is algebraically compact. As we know that  $x \notin V$  we have that  $f(x) \neq b$ . The topology on  $B$  is Hausdorff so there is an open affine cofinite set  $U$  with  $b \in U$  and  $f(x) \notin U$ .

However by Lemma 4.3.3 the set  $U$  will also be closed. Now  $f$  is continuous so the preimage  $f^{-1}(U)$  will be affine cofinite closed and we have

$$V = f^{-1}(b) \subset f^{-1}(U)$$

and  $x \notin f^{-1}(U)$ , so we let  $V_x$  be  $f^{-1}(U)$ .

We now have,

$$V = \bigcap_{x \notin V} V_x$$

that is,  $V$  is an intersection of affine cofinite closed sets. □

With the above lemma we can simplify the third condition of algebraic compactness.

**Lemma 4.3.11.** *In an algebraically compact algebra any family of closed affine sets with the finite intersection property has non-empty intersection.*

*Proof.* Suppose for some indexing set  $\lambda$  we have a family  $(S_i)_{i \in \lambda}$  of closed affine sets with the finite intersection property. By Lemma 4.3.10 for each  $i \in \lambda$  we have an indexing set  $\lambda_i$  and a family of affine cofinite closed sets  $(S_j^i)_{j \in \lambda_i}$  such that

$$S_i = \bigcap_{j \in \lambda_i} S_j^i.$$

For any  $i \in \lambda$  and  $j \in \lambda_i$  we have  $S_i \subset S_j^i$ . Given a finite indexing set  $I \subset \lambda$  and a family of finite indexing sets  $(I_i \subset \lambda_i)_{i \in I}$  we have

$$\bigcap_{i \in I} S_i \subset \bigcap_{\substack{i \in I \\ j \in I_i}} S_j^i.$$

However, by assumption, the sets  $(S_i)_{i \in \lambda}$  have the finite intersection property, so the left hand intersection is non-empty which means that the right hand intersection is non-empty.

This gives us that the family  $(S_j^i)_{j \in \lambda_i, i \in \lambda}$  has the finite intersection property. By algebraic compactness this family has non-empty intersection so we have

$$\bigcap_{i \in \lambda} S_i = \bigcap_{\substack{j \in \lambda_i \\ i \in \lambda}} S_j^i \neq \emptyset.$$

□

So, we have seen that our algebraically compact spaces will have a basis of clopen affine cofinite sets and will satisfy the stronger compactness condition given in Lemma 4.3.11.

## 4.4 Ultrafilters on Algebraically Compact Algebras

As we have seen, compact Hausdorff spaces are exactly those topological spaces on which ultrafilters always have a unique limit. Now we will consider ultrafilters on algebraically compact

algebras and we will prove an analogous result. First we define the notion of the limit of an ultrafilter on an algebra.

**Definition 4.4.1.** A *limit* of an ultrafilter  $\mathcal{U}$  on an algebra  $A$  is a point  $p \in A$  that is in the intersection of the closed sets of  $\mathcal{U}$ .

In other words, the limit of an ultrafilter  $\mathcal{U}$  on an algebra  $A$  is a point  $p \in A$  satisfying

$$T \in \mathcal{U} \Rightarrow p \in T$$

for all closed affine cofinite subsets  $T$  of  $A$ .

Suppose we have an ultrafilter  $\mathcal{U}$  on an algebra  $A$ . In the **Set** case we know that the closed sets are exactly the complements of the open sets. We used this fact to prove Lemma 4.1.2, that is, that a point is in the intersection of the closed sets of an ultrafilter if and only if every open set containing that point is in the ultrafilter.

The complement of an affine cofinite subset is not necessarily affine cofinite, so we cannot use the same argument to prove this fact in the general case. However, as we saw in Proposition 4.3.9, an affine cofinite set in an algebraically compact algebra is open if and only if it is closed. We will now see how we can use this fact to generalise Lemma 4.1.2.

**Lemma 4.4.2.** Given an ultrafilter  $\mathcal{U}$  on an algebraically compact algebra  $A$ ,  $p$  is a limit for  $\mathcal{U}$  if and only if every open affine cofinite subset of  $A$  containing  $p$  is in  $\mathcal{U}$ .

*Proof.* Suppose that  $p$  is a limit of  $\mathcal{U}$  and  $T$  is an open affine cofinite set with  $p \in T$ . Let  $T'$  be the coset of  $T$  that is in  $\mathcal{U}$ , as multiplication is continuous  $T'$  is also open. So, by Proposition 4.3.9,  $T'$  is closed, which gives  $T' \in \mathcal{C}(\mathcal{U})$ . However we have

$$p \in \bigcap_{S \in \mathcal{C}(\mathcal{U})} S \subset T'$$

so,  $T'$  is the coset of  $T$  containing  $p$ , so  $T = T' \in \mathcal{U}$ .

Now suppose that every open affine cofinite set containing  $p$  is in  $\mathcal{U}$ . Let  $R$  be a closed affine cofinite set with  $R \in \mathcal{U}$ . Let  $R'$  be the coset of  $R$  with  $p \in R'$ . As multiplication is continuous  $R'$  is also closed but then, by Lemma 4.3.8,  $R'$  is open.

So by assumption, as  $R'$  is an open affine cofinite set with  $p \in R'$  we have  $R' \in \mathcal{U}$ . However  $\mathcal{U}$  is an ultrafilter, so we must have  $R = R'$ , which gives  $p \in R$ .  $\square$

So, given an ultrafilter  $\mathcal{U}$  on the algebra  $A$  the point  $p$  is a limit of  $\mathcal{U}$  if, for all open affine cofinite subsets  $S$  of  $A$ , we have

$$p \in S \Rightarrow S \in \mathcal{U}.$$

**Proposition 4.4.3.** Suppose  $\mathcal{U}$  is an ultrafilter on an algebraically compact algebra, then  $p$  is a limit of  $\mathcal{U}$  if and only if, for all cofinite clopen subsets  $S$  of  $A$  we have

$$p \in S \Leftrightarrow S \in \mathcal{U}.$$

*Proof.* From Proposition 4.3.9 we have that an affine cofinite subset of an algebraically compact algebra is open if and only if it is closed. So by combining Definition 4.4.1 with Lemma 4.4.2 we get the above result.  $\square$



With Lemma 4.1.4 we showed that every ultrafilter on a compact Hausdorff space has a unique limit, we now generalise this result to algebras.

**Lemma 4.4.4.** *If  $\mathcal{U}$  is an ultrafilter on an algebraically compact algebra, then  $\mathcal{U}$  has a unique limit.*

*Proof.* Consider the following intersection,

$$L = \bigcap_{S \in \mathcal{O}(\mathcal{U})} S = \bigcap_{S \in \mathcal{C}(\mathcal{U})} S.$$

The family  $\mathcal{C}(\mathcal{U})$  has the finite intersection property so, by the third condition of algebraic compactness, must have non-empty intersection, so we have  $L \neq \emptyset$ .

Suppose we have two points  $p$  and  $q$  in  $L$ . By Hausdorffness there are open affine cofinite sets  $U_p$  and  $U_q$  with  $p \in U_p$  and  $q \in U_q$  such that  $U_p \cap U_q = \emptyset$ . By continuity of the group operation we have that all the cosets of  $U_p$  are open. As  $\mathcal{U}$  is an ultrafilter, one of these cosets,  $U'_p$ , is in  $\mathcal{U}$ . We also have

$$p \in L \subset U'_p.$$

However, we have  $p \in U_p$  and  $p \in U'_p$ , which gives

$$U_p = U'_p \in \mathcal{U}.$$

Similarly we have  $U_q \in \mathcal{U}$ , however this gives

$$\emptyset = U_p \cap U_q \in \mathcal{U},$$

a contradiction, so there is only one element in  $L$ , that is,  $\mathcal{U}$  has a unique limit. □

We now know that limits of ultrafilters on algebraically compact algebras are unique. So given an ultrafilter  $\mathcal{U}$  on an algebraically compact algebra we denote the limit of  $\mathcal{U}$  by  $\lim(\mathcal{U})$ .

**Proposition 4.4.5.** *If  $\mathcal{U}$  is an ultrafilter on an algebraically compact algebra, then  $\lim(\mathcal{U})$  is the unique point satisfying*

$$T \in \mathcal{U} \Leftrightarrow \lim(\mathcal{U}) \in T$$

*for all clopen affine cofinite sets.*

*Proof.* This follows from Proposition 4.4.3 and Lemma 4.4.4. □

## Chapter 5

# The Codensity Monad

In this chapter we first review the notion of codensity. The material of this chapter is standard, we develop it here as the details will be necessary in the following chapters. Dense and codense subcategories were originally defined by Isbell in [9]. Whereas Isbell used the terminology left adequate and right adequate subcategory, the terms dense and codense were introduced by Ulmer in [22]. In this paper Ulmer also generalised the notion of density to arbitrary functors.

Not every subcategory is codense, with certain limited assumptions, those that aren't induce a monad referred to as the codensity monad. The codensity monad was introduced independently by Kock in [11] and Appelgate and Tierney in [2]. In the second section of this chapter we explicitly construct the endofunctor, unit and multiplication morphisms of the codensity monad and in the third section we prove that they satisfy the monad axioms. In the next chapter, given a category of algebras we will consider the codensity monad induced by the subcategory of finite algebras.

### 5.1 Codensity

We begin by discussing the notion of codensity. Suppose we have a category  $\mathcal{C}$  with small limits and an object  $C$  of  $\mathcal{C}$ . Following the notation of MacLane in [14] we let  $(C \downarrow \mathcal{C})$  be the usual coslice category and let  $U_C$  be the forgetful functor from  $(C \downarrow \mathcal{C})$  to  $\mathcal{C}$ . Consider the diagram

$$(C \downarrow \mathcal{C}) \xrightarrow{U_C} \mathcal{C}$$

in  $\mathcal{C}$ . Intuitively, if we think of this diagram as a subcategory of  $\mathcal{C}$ , the objects of this diagram are the codomains of morphisms with domain  $C$  and the morphisms of this diagram are morphisms  $B \xrightarrow{h} B'$  in  $\mathcal{C}$  that satisfy commutative diagrams of the form

$$\begin{array}{ccc} & C & \\ f \swarrow & & \searrow f \\ B & \xrightarrow{h} & B' \end{array}$$

There is a canonical cone from  $C$  to this diagram, this cone is in fact just the family of morphisms with domain  $C$ . To see this, first suppose we have an object  $C \xrightarrow{f} B$  in  $(C \downarrow \mathcal{C})$ , we

have  $U_C(f) = B$  so we have a morphism

$$C \xrightarrow{f} U_C(f)$$

in  $\mathcal{C}$ . Now suppose we have a morphism in  $(C \downarrow \mathcal{C})$ , that is a commutative diagram of the form

$$\begin{array}{ccc} & C & \\ f \swarrow & & \searrow f' \\ B & \xrightarrow{h} & B' \end{array}$$

then the diagram

$$\begin{array}{ccc} & C & \\ f \swarrow & & \searrow f' \\ U_C(f) & \xrightarrow{U_C h} & U_C(f') \end{array}$$

commutes. So the family of morphisms with domain  $C$  form a cone from  $C$  to the diagram

$$(C \downarrow \mathcal{C}) \xrightarrow{U_{\mathcal{C}}} \mathcal{C}.$$

Suppose now we have some category  $\mathcal{D}$  and a functor  $F$  from  $\mathcal{D}$  to  $\mathcal{C}$ . Given an object  $C$  of  $\mathcal{C}$ , we let  $(C \downarrow F)$  be the usual coslice category and  $U_{(C \downarrow F)}$  be the forgetful functor from  $(C \downarrow F)$  to  $\mathcal{D}$ . Now, consider the diagram

$$(C \downarrow F) \xrightarrow{U_{(C \downarrow F)}} \mathcal{D} \xrightarrow{F} \mathcal{C}.$$

If we think of this diagram as a subcategory of  $\mathcal{C}$ , the objects of this subcategory will be the codomains of the morphisms in  $\mathcal{C}$  of the form  $C \xrightarrow{f} FD$  for  $D$  in  $\mathcal{D}$  and the morphisms of this subcategory will be morphisms  $FD \xrightarrow{Fh} FD'$  in  $\mathcal{C}$  that satisfy commutative diagrams of the form

$$\begin{array}{ccc} & C & \\ f \swarrow & & \searrow f' \\ FD & \xrightarrow{Fh} & FD' \end{array}$$

By the same construction as above, there is a canonical cone from  $C$  to this diagram, this cone consists of all morphisms with domain  $C$  and codomain of the form  $FD$  for some  $D$  in  $\mathcal{D}$ .

**Definition 5.1.1.** We say that a functor  $F : \mathcal{D} \rightarrow \mathcal{C}$  is **codense** if for every  $C$  in  $\mathcal{C}$  the canonical cone under  $C$  described above is a limit cone.

In particular if the functor  $F : \mathcal{D} \rightarrow \mathcal{C}$  is codense then for every object  $C$  of  $\mathcal{C}$  we have

$$C \cong \lim((C \downarrow F) \xrightarrow{U_{(C \downarrow F)}} \mathcal{D} \xrightarrow{F} \mathcal{C}).$$

**Definition 5.1.2.** Given a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , we define functor  $\mathcal{C}(-, F)$  from  $\mathcal{C}$  to  $[\mathcal{D}, \text{Set}]^{op}$  by

$$\mathcal{C}(-, F)(C') = \mathcal{C}(C', F-)$$

for an object  $C'$  in  $\mathcal{C}$  and

$$\mathcal{C}(-, F)(f) = \mathcal{C}(f, F-)$$

for a morphism  $C \xrightarrow{f} C'$  in  $\mathcal{C}$ .

In the following lemma we see how we can reformulate codensity in terms of the functor defined above.

**Lemma 5.1.3.** *A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is codense if and only if the functor  $\mathcal{C}(-, F)$  is full and faithful.*

*Proof.* To say that  $C$  is the limit of its canonical cone over the diagram  $FU_{(C \downarrow F)}$  is to say that for every  $C'$  in  $\mathcal{C}$  a cone from  $C'$  to  $FU_{(C \downarrow F)}$  factorises uniquely through  $C$ , that is, we have a natural isomorphism

$$\mathcal{C}(C', C) \cong \text{Cone}(C', FU_{(C \downarrow F)}),$$

where  $\text{Cone}(A, G)$  is the set of cones from the object  $A$  to the diagram  $G$ .

Now suppose that  $\tau$  is a cone from  $C'$  to the diagram  $FU_{(C \downarrow F)}$ . So given a morphism  $C \xrightarrow{g} FD$  we have a morphism  $C' \xrightarrow{\tau_g} FD$  such that for any morphism  $D \xrightarrow{f} D'$  in  $\mathcal{D}$  the triangle

$$\begin{array}{ccc} C' & \xrightarrow{\tau_g} & FD \\ & \searrow \tau_{Ffg} & \downarrow Ff \\ & & FD' \end{array}$$

commutes. Given  $D$  in  $\mathcal{D}$  we let  $\tau_D$  be the morphism from  $\mathcal{C}(C, FD)$  to  $\mathcal{C}(C', FD)$  given by

$$\tau_D(f) = \tau_f$$

for all morphisms  $f$  in  $\mathcal{C}(C, FD)$ . Given a morphism  $D \xrightarrow{f} D'$  in  $\mathcal{D}$  the above triangle commutes if and only if the square

$$\begin{array}{ccc} \mathcal{C}(C, FD) & \xrightarrow{\tau_D} & \mathcal{C}(C', FD) \\ \mathcal{C}(C, Ff) \downarrow & & \downarrow \mathcal{C}(C', Ff) \\ \mathcal{C}(C, FD') & \xrightarrow{\tau_{D'}} & \mathcal{C}(C', FD') \end{array}$$

commutes. Which is to say that  $\tau$  is a natural transformation from  $\mathcal{C}(C, F-)$  to  $\mathcal{C}(C', F-)$ .

So a cone from  $C'$  to the diagram  $FU_{(C \downarrow F)}$  is equivalent to a morphism in  $[\mathcal{D}, \text{Set}]^{op}$  from  $\mathcal{C}(C, F)$  to  $\mathcal{C}(C', F)$

This means that  $C$  is the limit of its canonical cone over  $FU_{(C \downarrow F)}$  if and only if for all  $C'$  in  $\mathcal{C}$  we have a natural isomorphism

$$\text{Cone}(C', FU_{(C \downarrow F)}) \cong [\mathcal{D}, \text{Set}]^{op}(\mathcal{C}(C', F), \mathcal{C}(C, F)).$$

Combining this with our previous natural isomorphism we have that  $C$  is the limit of its canonical cone over  $FU_{(C \downarrow F)}$  if and only if

$$\mathcal{C}(C', C) \cong [\mathcal{D}, \text{Set}]^{op}(\mathcal{C}(C', F), \mathcal{C}(C, F)).$$

However this is equivalent to the functor  $\mathcal{C}(-, F)$  being full and faithful. So, putting this all

together, the functor  $F$  is codense if and only if the functor  $\mathcal{C}(-, F)$  is full and faithful.  $\square$

Given a set  $S$  and an object  $C$  of  $\mathcal{C}$  we write  $[S, C]$  for the product

$$\prod_{s \in S} C.$$

We can rewrite the limit in the definition of codensity as an end as follows

$$\lim((C \downarrow F) \xrightarrow{U_{(C \downarrow F)}} \mathcal{D} \xrightarrow{F} \mathcal{C}) = \int_{D \in \mathcal{D}} [\mathcal{C}(C, FD), FD].$$

So the the functor  $F$  is codense if for every object  $C$  of  $\mathcal{C}$  we have

$$C \cong \int_{D \in \mathcal{D}} [\mathcal{C}(C, FD), FD]$$

which is to say that the canonical cone under  $C$  is the limiting cone. This is the formulation of codensity that we rely upon most in the sequel.

The end formula above is in fact the object function of the right Kan extension of  $F$  along itself. Indeed we can rephrase the definition of codensity in terms of Kan extensions, we will see this in detail in section 5.2.2. The functor  $F$  is codense if the right Kan extension of  $F$  along itself is the identity functor  $1_{\mathcal{C}}$  with the identity transformation  $F \xrightarrow{1_F} F$ .

## 5.2 The Endofunctor $T$

### 5.2.1 Definition and Properties of $T$

Suppose again that we have categories  $\mathcal{D}$  and  $\mathcal{C}$ , that  $\mathcal{C}$  has all small limits and that we have a functor  $F$  from  $\mathcal{D}$  to  $\mathcal{C}$ . Even if this functor is not codense, we can still consider the diagram

$$(C \downarrow F) \xrightarrow{U_{(C \downarrow F)}} \mathcal{D} \xrightarrow{F} \mathcal{C}$$

as above and take its limit.

**Definition 5.2.1.** *The endofunctor  $T^F$  of  $\mathcal{C}$  is defined for an object  $C$  of  $\mathcal{C}$  by*

$$T^F C = \int_{D \in \mathcal{D}} [\mathcal{C}(C, FD), FD]$$

and for a morphism  $C \xrightarrow{g} C'$  in  $\mathcal{C}$  by

$$T^F g = \int_{D \in \mathcal{D}} [\mathcal{C}(g, FD), FD].$$

The functoriality of the limit ensures that  $T^F$  will respect composition and identities and so will be a functor. As we are dealing with a fixed functor  $F : \mathcal{D} \rightarrow \mathcal{C}$ , henceforth we write  $T$  for  $T^F$ . We have defined the object  $TC$  above as an end, so there is an associated canonical cone.

**Definition 5.2.2.** Given objects  $C$  of  $\mathcal{C}$  and  $D$  of  $\mathcal{D}$  and a morphism  $C \xrightarrow{f} FD$  in  $\mathcal{C}$ , then  $\pi_f$  is the morphism from  $TC$  to  $F(U_{(C \downarrow F)}(f))$  of the associated canonical cone.

The two useful lemmas below follow from the definition of the functor  $T$  as a limit.

**Lemma 5.2.3.** Given a morphism  $C \xrightarrow{f} FD$  in  $\mathcal{C}$  and a morphism  $D \xrightarrow{h} D'$  in  $\mathcal{D}$  we have

$$Fh \circ \pi_f = \pi_{Fhf}.$$

*Proof.* Consider the diagram

$$\begin{array}{ccc} & C & \\ f \swarrow & & \searrow Fhf \\ FD & \xrightarrow{Fh} & FD' \end{array}$$

this will be a morphism in the slice category  $(C \downarrow F)$ . The object  $TC$  is the limit

$$\lim((C \downarrow F) \xrightarrow{U} \mathcal{D} \xrightarrow{F} \mathcal{C})$$

so the diagram

$$\begin{array}{ccc} & TC & \\ \pi_f \swarrow & & \searrow \pi_{Fhf} \\ FD & \xrightarrow{Fh} & FD' \end{array}$$

commutes. That is we have

$$Fh \circ \pi_f = \pi_{Fhf}.$$

□

**Lemma 5.2.4.** Given an object  $D$  of  $\mathcal{D}$ , objects  $C$  and  $C'$  of  $\mathcal{C}$  and morphisms  $C \xrightarrow{g} C'$  and  $C' \xrightarrow{f} FD$  in  $\mathcal{C}$  we have

$$\begin{array}{ccc} TC & \xrightarrow{Tg} & TC' \\ & \searrow \pi_{fg} & \downarrow \pi_f \\ & & FD \end{array}$$

*Proof.* We have that the morphism

$$\mathcal{C}(C', FD) \xrightarrow{\mathcal{C}(g, FD)} \mathcal{C}(C, FD)$$

sends  $f$  to  $fg$ . So, as it is a limit, the diagram

$$\begin{array}{ccc} \prod_{\mathcal{C}(C, FD)} FD & \xrightarrow{\prod_{\mathcal{C}(g, FD)} FD} & \prod_{\mathcal{C}(C', FD)} FD \\ & \searrow & \swarrow \\ & FD & \end{array}$$

commutes, where the unlabeled arrows are the canonical projection morphisms. This gives that

the diagram

$$\begin{array}{ccc} TC & \xrightarrow{Tg} & TC' \\ & \searrow \pi_{fg} & \downarrow \pi_f \\ & & FD \end{array}$$

commutes □

As outlined by MacLane in section IX.5 of [14] we can write the set of natural transformations between two functors as an end. Specifically, for any pair of functors  $G$  and  $H$  from  $\mathcal{D}$  to  $\mathcal{C}$  we have

$$\text{Nat}(G, H) = \int_{D \in \mathcal{D}} \mathcal{C}(GD, HD).$$

So, given an object  $C$  of  $\mathcal{C}$ , if we let  $G = \mathcal{C}(C, F-)$  and  $H = F$ , we have

$$TC = \text{Nat}(\mathcal{C}(C, F-), F).$$

For most of this thesis we consider categories of algebras where our objects are sets with some algebraic structure. For the rest of this subsection, let us suppose that this is the case with  $\mathcal{C}$ .

Suppose that we have an element  $I$  of  $TC$ . By the above, we can think of  $I$  as a natural transformation from  $\mathcal{C}(C, F-)$  to  $F$ . That is, given an object  $D$  of  $\mathcal{D}$  we have a morphism  $I_D$  that sends a morphism  $C \xrightarrow{f} FD$  to an element  $I_D(f)$  of  $D$  such that, if we have a morphism  $FD \xrightarrow{h} FD'$  then we have  $I_{D'}(hf) = hI_D(f)$ . Indeed the naturality of elements of  $TA$  is equivalent to Lemma 5.2.3.

Thinking of elements of  $TC$  as the natural transformations described above, if we have a morphism  $C \xrightarrow{f} FD$  then the projection morphism  $\pi_f$  from  $TC$  to  $FD$  is given by

$$\pi_f(I) = I_D(f)$$

for  $I$  in  $TC$ .

Suppose we have a morphism  $C \xrightarrow{g} C'$  in  $\mathcal{C}$ , thinking again of the elements of  $TC$  as the natural transformations described above, we can understand  $Tg$  particularly easily, we have

$$Tg = \text{Nat}(\mathcal{C}(g, F-), F-)$$

so, given  $I$  in  $TC$  and a morphism  $C' \xrightarrow{f} FD$ , we have

$$Tg(I)(f) = I_D(fg).$$

We will rely both on the limit and natural transformation understanding of  $T$  in the sequel.

For  $C$  an object of  $\mathcal{C}$ ,  $TC$  is an algebra, so we can ask how the operations of our theory are defined on  $TC$ . Let,  $|-|$  be the forgetful functor from  $\mathcal{C}$  to **Set**. If  $\lambda$  is an  $n$ -ary operation for our theory then it is a natural transformation from  $|-|^n$  to  $|-|$ . So, given an object  $D$  of  $\mathcal{D}$

and a morphism  $C \xrightarrow{f} FD$ , by naturality we will have

$$\begin{array}{ccc} |TC| \times \dots \times |TC| & \xrightarrow{|\pi_f| \times \dots \times |\pi_f|} & |FD| \times \dots \times |FD| \\ \lambda_{TC} \downarrow & & \downarrow \lambda_{FD} \\ |TC| & \xrightarrow{|\pi_f|} & |FD| \end{array}$$

So given a family  $(I_i)_{1 \leq i \leq n}$  of elements of  $TC$  we have

$$\lambda_{TC}(I_1, \dots, I_n)(f) = \lambda_{FD}(I_1(f), \dots, I_n(f))$$

that is, the operations are defined pointwise on  $TC$ .

There are certain elements of  $TC$  that are very simple to describe. Given an element  $c$  of  $C$ , we let  $ev_c$  be the evaluation morphism that sends  $C \xrightarrow{f} FD$  to  $f(c)$ .

**Lemma 5.2.5.** *For  $c$  in  $C$ ,  $ev_c$  is an element of  $TC$ .*

*Proof.* Given morphisms  $C \xrightarrow{f} FD$  and  $D \xrightarrow{h} D'$  in  $\mathcal{D}$ , we have

$$ev_c(Fhf) = Fh(f(c)) = Fh(ev_c(f)).$$

So  $ev_c$  is a natural transformation from  $\mathcal{C}(C, F-)$  to  $F$ , so it is an element of  $TC$ .  $\square$

## 5.2.2 $T$ as a Kan Extension

We now show that  $T$  can be constructed as a Kan extension. We define a class of distinguished projection morphisms and show that they form a natural transformation.

**Definition 5.2.6.** *Given an object  $D$  of  $\mathcal{D}$ , we define  $\pi_D$  as the projection morphism  $\pi_{1_{FD}}$  from  $TFD$  to  $FD$ .*

**Lemma 5.2.7.** *The family of morphisms,  $\pi_D$ , is a natural transformation from  $TF$  to  $F$ .*

*Proof.* Given a morphism  $D \xrightarrow{h} D'$  in  $\mathcal{D}$  consider the diagram

$$\begin{array}{ccc} TFD & \xrightarrow{\pi_{1_{FD}}} & FD \\ TFh \downarrow & \searrow \pi_{Fh} & \downarrow Fh \\ TFD' & \xrightarrow{\pi_{1_{FD'}}} & FD' \end{array}$$

The bottom left triangle commutes by Lemma 5.2.4 and the upper right triangle commutes by Lemma 5.2.3. So the outer square commutes, however this is just the square

$$\begin{array}{ccc} TFD & \xrightarrow{\pi_D} & FD \\ TFh \downarrow & & \downarrow Fh \\ TFD' & \xrightarrow{\pi_{D'}} & FD' \end{array}$$



so  $\pi$  is a natural transformation. □

We will now see that the endofunctor  $T$  along with the natural transformation  $\pi$  is the right Kan extension of the functor  $F$  along itself. That is, given an endofunctor  $S$  of  $\mathcal{C}$  and a natural transformation  $\delta$  from  $SF$  to  $F$  as below

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{F} & \mathcal{C} \\ & \searrow F & \downarrow S \\ & & \mathcal{C} \end{array} \quad \begin{array}{c} \swarrow \delta \\ \downarrow \end{array}$$

there is a unique natural transformation  $\epsilon$  from  $S$  to  $T$  such that

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{F} & \mathcal{C} \\ & \searrow F & \downarrow T \\ & & \mathcal{C} \end{array} \quad \begin{array}{c} \swarrow \pi \\ \downarrow \end{array} \quad \begin{array}{c} \swarrow \epsilon \\ \downarrow \end{array} \quad S \quad \text{equals} \quad \begin{array}{ccc} \mathcal{D} & \xrightarrow{F} & \mathcal{C} \\ & \searrow F & \downarrow S \\ & & \mathcal{C} \end{array} \quad \begin{array}{c} \swarrow \delta \\ \downarrow \end{array}$$

Suppose we have an endofunctor  $S$  and a natural transformation  $\delta$ , as above, we will construct a natural transformation  $\epsilon$  from  $S$  to  $T$  that is the unique such transformation satisfying  $\pi \circ \epsilon F = 1_F$ . The endofunctor  $T$  is defined as a limit and we will use the universal property of this limit to construct the natural transformation  $\epsilon$ . To do this we first construct a cone over the diagram of  $TC$ , for  $C$  an object of  $\mathcal{C}$ .

**Lemma 5.2.8.** *Given an object  $C$  of  $\mathcal{C}$ , the family of morphisms  $(\delta_D S f)_{C \downarrow F D}$  forms a cone from  $SC$  to the diagram*

$$(C \downarrow F) \xrightarrow{U_{(C \downarrow F)}} \mathcal{D} \xrightarrow{F} \mathcal{C}$$

in  $\mathcal{C}$ .

*Proof.* Suppose we have a morphism  $h$  in  $(C \downarrow F)$  from  $C \xrightarrow{f} FD$  to  $C \xrightarrow{g} FD'$ , that is, we have a commutative diagram of the form

$$\begin{array}{ccc} & C & \\ f \swarrow & & \searrow g \\ FD & \xrightarrow{Fh} & FD' \end{array}$$

Now consider the diagram

$$\begin{array}{ccc} & SC & \\ Sf \swarrow & & \searrow Sg \\ SFD & \xrightarrow{SFh} & SFD' \\ \delta_D \downarrow & & \downarrow \delta_{D'} \\ FD & \xrightarrow{Fh} & FD' \end{array}$$

The top triangle commutes as it is  $S$  applied to the previous diagram and the bottom square

commutes as  $\delta$  is a natural transformation, so the outer pentagram commutes, that is

$$\begin{array}{ccc} & SC & \\ \delta_D Sf \swarrow & & \searrow \delta_{D'} Sg \\ FD & \xrightarrow{Fh} & FD' \end{array}$$

commutes. So  $(\delta_D Sf)_{C \xrightarrow{f} FD}$  forms a cone from  $SC$  to the diagram

$$(C \downarrow F) \xrightarrow{U_{(C \downarrow F)}} \mathcal{D} \xrightarrow{F} \mathcal{C}.$$

□

The object  $TC$  is the limit of the diagram  $FU_{(C \downarrow F)}$ , so we can make the following definition.

**Definition 5.2.9.** *Given  $S$  and  $\delta$  as above, we define  $\epsilon_C$  as the unique morphism such that the diagram*

$$\begin{array}{ccc} SC & \xrightarrow{Sf} & SFD \\ \epsilon_A \downarrow & & \downarrow \delta_D \\ TC & \xrightarrow{\pi_f} & FD \end{array}$$

*commutes for every morphism  $C \xrightarrow{f} FD$  in  $\mathcal{C}$ .*

Suppose we have an object  $C$  of  $\mathcal{C}$  a morphism  $C \xrightarrow{f} FD$  in  $\mathcal{C}$ . Suppose also that our category  $\mathcal{C}$  is some category of algebras. Thinking of the elements of  $SC$  and  $TC$  as natural transformations, the above lemma states that, for  $c$  an object of  $C$ ,  $\epsilon_C$  is the morphism given by  $\epsilon_C(c)(f) = \delta_D(Sf(c))$ .

**Lemma 5.2.10.** *The family of morphisms  $\epsilon_C$ , for  $C$  object of  $\mathcal{C}$ , is a natural transformation from  $S$  to  $T$ .*

*Proof.* Suppose we have a morphism,  $C \xrightarrow{h} C'$  in  $\mathcal{C}$ . Then, given another morphism and  $C' \xrightarrow{f} FD$  in  $\mathcal{C}$  consider the diagram

$$\begin{array}{ccccc} SC & \xrightarrow{Sh} & SC' & \xrightarrow{Sf} & SFD \\ \epsilon_C \downarrow & & \downarrow \epsilon_{C'} & & \downarrow \delta_D \\ TC & \xrightarrow{Th} & TC' & \xrightarrow{\pi_f} & FD \end{array}$$

The right hand square commutes by the definition of  $\epsilon$ . By Lemma 5.2.4 and as  $S$  is a functor the outer square is equal to

$$\begin{array}{ccc} SC & \xrightarrow{Sfh} & SFD \\ \epsilon_A \downarrow & & \downarrow \delta_D \\ TC & \xrightarrow{\pi_{fh}} & FD \end{array}$$

which also commutes by the definition of  $\epsilon$ . So the diagram

$$\begin{array}{ccccc} SC & \xrightarrow{Sh} & TC' & & \\ \epsilon_C \downarrow & & \downarrow \epsilon_{C'} & & \\ TC & \xrightarrow{Th} & TC' & \xrightarrow{\pi_f} & FD \end{array}$$

commutes. Now, the family of morphisms  $\pi_f$  for all  $f$  in  $\mathcal{C}$  of the form  $C' \xrightarrow{f} FD$  are the projection morphisms for a limit, so the family of morphisms  $\pi_f \circ Th \circ \epsilon_C$  are a cone for this limit.

This means that there is a unique morphism  $e$  from  $SC$  to  $TC'$  such that, for all morphisms  $C' \xrightarrow{f} FD$  in  $\mathcal{C}$ , the diagram

$$\begin{array}{ccccc} SC & & & & \\ \epsilon_C \downarrow & \searrow e & & & \\ TC & \xrightarrow{Th} & TC' & \xrightarrow{\pi_f} & FD \end{array}$$

commutes. However we have that both  $Th \circ \epsilon_C$  and  $\epsilon_{C'} \circ Sh$  satisfy this condition, so, by uniqueness, they must be equal.

So the diagram

$$\begin{array}{ccc} SC & \xrightarrow{Sh} & SC' \\ \epsilon_C \downarrow & & \downarrow \epsilon_{C'} \\ TC & \xrightarrow{Th} & TC' \end{array}$$

commutes. That is,  $\epsilon$  is a natural transformation. □

For  $D$  an object of  $\mathcal{D}$ , the morphism  $\pi_D$  is defined as  $\pi_{1_{FD}}$ . So, by the definition of  $\epsilon$ , we have

$$\begin{array}{ccc} SFD & \xrightarrow{S1_{FD}} & SD \\ \epsilon_{FD} \downarrow & & \downarrow \delta_D \\ TFD & \xrightarrow{\pi} & D \end{array}$$

that is

$$\pi \circ \epsilon_F = \delta.$$

To show that  $T$  is a Kan extension we need to show that  $\epsilon$  is the unique morphism with the above property.

**Lemma 5.2.11.** *The natural transformation  $\epsilon$  from  $S$  to  $T$  is the unique such natural transformation that satisfies*

$$\pi \circ \epsilon_F = \delta.$$

*Proof.* For any natural transformation  $\tau$  from  $S$  to  $T$ , given a morphism  $C \xrightarrow{f} FD$  in  $\mathcal{C}$ , consider

the diagram

$$\begin{array}{ccccc}
SC & \xrightarrow{\tau_C} & TC & & \\
Sf \downarrow & & Tf \downarrow & \searrow \pi_f & \\
SFD & \xrightarrow{\tau_{FD}} & TFD & \xrightarrow{\pi_D} & FD
\end{array}$$

The left square commutes as  $\tau$  is a natural transformation and the right triangle commutes by the definition of  $\pi$  and Lemma 5.2.4 so the outermost square commutes.

Now suppose we have a natural transformation  $\epsilon'$  from  $S$  to  $T$  with  $\pi \circ \epsilon' = \delta$ . Consider the diagram

$$\begin{array}{ccccc}
& & SC & & \\
& \epsilon_C \swarrow & \downarrow Sf & \searrow \epsilon'_C & \\
TC & & SFD & & TC \\
& \epsilon_{FD} \swarrow & \downarrow \delta_D & \searrow \epsilon'_{FD} & \\
TFD & & & & TFD \\
& \pi_D \swarrow & \downarrow & \searrow \pi_D & \\
& & FD & & 
\end{array}$$

The leftmost and rightmost pentagons are examples of the previous diagram with  $\epsilon$  and  $\epsilon'$  for  $\tau$ , so they commute. The left interior triangle commutes by the remark preceding this lemma and the right interior triangle commutes by assumption so the interior square commutes.

Putting this together, the outer square commutes, that is,  $\pi_f \circ \epsilon_C = \pi_f \circ \epsilon'_C$ . Using the same argument as that used in Lemma 5.2.10, the  $\pi_f$  are projection morphisms for a limit and so  $\pi_f \circ \epsilon_C$  is a cone for this limit. So there is a unique morphism  $e$  from  $SFD$  to  $TFD$  such that, for all morphisms  $C \xrightarrow{f} FD$  in  $\mathcal{C}$ , we have  $\pi_f \circ e = \pi_f \circ \epsilon_C$ .

However both  $\epsilon_C$  and  $\epsilon'_C$  satisfy this condition so, by uniqueness, we have  $\epsilon_C = \epsilon'_C$ , that is,  $\epsilon$  is the unique morphism such that  $\pi \circ \epsilon = \delta$ .  $\square$

So, combining the above we have the following result.

**Proposition 5.2.12.** *The endofunctor  $T$  along with the natural transformation  $\pi$  is the right Kan extension of  $F$  along itself.*

*Proof.* This follows directly from Definition 5.2.7 and Lemmas 5.2.10 and 5.2.11.  $\square$

## 5.3 The Monad Structure

The endofunctor  $T$  on  $\mathcal{C}$  comes with a monad structure on  $\mathcal{C}$ . This monad is the codensity monad. In this section we construct this monad explicitly.

### 5.3.1 The Unit Morphism

We now define the morphism that will be the unit morphism of our monad. As we have seen, given an object  $C$  of  $\mathcal{C}$  there is a canonical cone from  $C$  to the diagram

$$(C \downarrow F) \xrightarrow{U} \mathcal{D} \xrightarrow{F} \mathcal{C}.$$

We have defined  $TC$  as the limit of this diagram. If  $F$  is codense then we have  $TC \cong C$ . However, if  $F$  is not codense there is still a canonical morphism from  $C$  to  $TC$ , which we define below.

**Definition 5.3.1.** *Given an object  $C$  of  $\mathcal{C}$ , we let  $C \xrightarrow{\eta_C} TC$  be the unique morphism from  $C$  to  $TC$  such that*

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & TC \\ & \searrow f & \downarrow \pi_f \\ & & FD \end{array}$$

*commutes for all morphisms  $C \xrightarrow{f} FD$  in  $\mathcal{C}$ .*

Given  $c$  in  $C$  and a morphism  $C \xrightarrow{f} FD$ , by the above definition we have that

$$\eta_C(c)(f) = f(c)$$

so for all objects  $c$  of  $C$  we have

$$\eta_C(c) = ev_c.$$

**Lemma 5.3.2.** *The family of morphisms  $C \xrightarrow{\eta_C} TC$ , for  $C$  an object of  $\mathcal{C}$ , define a natural transformation from  $1_{\mathcal{C}}$  to  $T$ .*

*Proof.* Suppose we have morphisms  $C \xrightarrow{f} C'$  and  $C' \xrightarrow{g} FD$  in  $\mathcal{C}$ , consider the diagram

$$\begin{array}{ccccc} C & \xrightarrow{\eta_C} & TC & & \\ & \searrow gf & \swarrow \pi_{gf} & & \\ & & FD & & \\ & \swarrow g & \nwarrow \pi_g & & \\ C' & \xrightarrow{\eta_{C'}} & TC' & \xrightarrow{Tf} & FD \end{array}$$

The upper and lower triangles commute by the definition of  $\eta$ . The right hand triangle commutes by Lemma 5.2.4 and the left hand triangle commutes by definition. Putting this together the diagram

$$\begin{array}{ccccc} C & \xrightarrow{\eta_C} & TC & & \\ f \downarrow & & \downarrow Tf & & \\ C' & \xrightarrow{\eta_{C'}} & TC' & \xrightarrow{\pi_g} & FD \end{array}$$

commutes.

However, the family of morphisms  $\pi_g$ , for  $g$  a morphism in  $\mathcal{C}$  of the form  $C' \xrightarrow{g} FD$ , are the projection morphisms for a limit. So, we have a cone  $\pi_g \circ \eta_{C'} \circ f$  for this limit. From this, we have that there is a unique morphism  $e$  from  $C$  to  $TC'$  such that the diagram

$$\begin{array}{ccccc} C & & & & \\ f \downarrow & \searrow e & & & \\ C' & \xrightarrow{\eta_{C'}} & TC' & \xrightarrow{\pi_g} & FD \end{array}$$

commutes.

However both  $Tf \circ \eta_C$  and  $\eta_{C'} \circ f$  satisfy this condition so, by uniqueness, the diagram

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & TC \\ f \downarrow & & \downarrow Tf \\ C' & \xrightarrow{\eta_{C'}} & TC' \end{array}$$

commutes, so  $\eta$  is a natural transformation.  $\square$

We will see in the proof of the following lemma that  $\eta$  is always split monic and under certain conditions is in fact an isomorphism.

**Lemma 5.3.3.** *If  $F$  is a full functor then  $TF$  and  $F$  are naturally isomorphic.*

*Proof.* Suppose we have some object  $D$  of  $\mathcal{D}$ . Consider the diagram

$$\begin{array}{ccc} FD & \xrightarrow{\eta_{FD}} & TFD \\ & \searrow 1_{FD} & \downarrow \pi_D \\ & & FD \end{array}$$

this commutes by the definition of  $\eta$  so we have  $\pi_D \circ \eta_{FD} = 1_{FD}$ .

In the other direction, suppose we have some morphism  $FD \xrightarrow{g} FD'$ . As  $F$  is a full functor there is a morphism  $D \xrightarrow{h} D'$  in  $\mathcal{D}$  such that  $Fh = g$ . Now consider the diagram

$$\begin{array}{ccc} TFD & & \\ \pi_D \downarrow & \searrow \pi_{Fh} & \\ FD & \xrightarrow{Fh} & FD' \\ \eta_{FD} \downarrow & \nearrow \pi_{Fh} & \\ TFD & & \end{array}$$

the upper triangle commutes by Lemma 5.2.3, the lower triangle commutes by the definition of  $\eta$  so the outer square commutes.

So, for all morphisms of the form  $FD \xrightarrow{g} FD'$  in  $\mathcal{C}$ , we have  $\pi_{g} \circ \eta_{FD} \circ \pi_D = \pi_{g'}$ . However as  $\pi_g$  are the projection morphisms for a limit, the family of morphisms  $\pi_g \circ \eta_{FD} \circ \pi_D$  are a cone for this limit.

So there is a unique morphism  $e$  from  $TFD$  to  $TFD$  such that we have  $\pi_g \circ \eta_{FD} \circ \pi_D = \pi_g \circ e$ . However both  $\eta_{FD} \circ \pi_D$  and  $1_{TFD}$  satisfy this condition. So, by uniqueness, we have  $\eta_{FD} \circ \pi_D = 1_{TFD}$ , so  $\eta_{FD}$  is an isomorphism, with inverse  $\pi_D$ .  $\square$

So, when  $F$  is a full functor, given an object  $D$  of  $\mathcal{D}$ , all the elements of  $TFD$  are of the form  $ev_d$  for some  $d$  in  $D$ .

### 5.3.2 The Multiplication Morphism

We now define the morphism that will be the multiplication morphism of our monad. For any morphism  $C \xrightarrow{f} FD$  we have the projection  $\pi_f$  from  $TC$  to  $D$ . In particular, for any morphism

$C \xrightarrow{f} FD$ , as  $\pi_f$  is a morphism from  $TC$  to  $D$  it induces the double projection morphism  $\pi_{\pi_f}$  from  $TTC$  to  $D$ . For convenience we introduce special notation for this double projection morphism below.

**Definition 5.3.4.** Given a morphism  $C \xrightarrow{f} FD$  in  $\mathcal{C}$ , we define  $\Pi_f$  to be the double projection morphism from  $TTC$  to  $FD$  induced by the projection morphism  $\pi_f$  from  $TC$  to  $FD$ .

The endofunctor  $T$  is defined as a limit and we will use the universal property of this limit to define the multiplication morphism of our monad. Given an object  $C$  of  $\mathcal{C}$  we will use the double projection morphisms defined above to construct a cone over the diagram of the limit  $TC$ .

**Lemma 5.3.5.** Given an object  $C$  of  $\mathcal{C}$ , the family of morphisms  $\Pi_f$ , for  $C \xrightarrow{f} FD$  a morphism in  $\mathcal{C}$ , will form a cone from  $TTC$  to the diagram

$$(C \downarrow F) \xrightarrow{U} \mathcal{D} \xrightarrow{F} \mathcal{C}.$$

*Proof.* Suppose we have a morphism  $h$  in the slice category  $(C \downarrow F)$  from  $C \xrightarrow{f} FD$  to  $C \xrightarrow{f'} FD'$ . That is, a commutative diagram in  $\mathcal{C}$  of the form

$$\begin{array}{ccc} & C & \\ f \swarrow & & \searrow f' \\ FD & \xrightarrow{Fh} & FD' \end{array}$$

By Lemma 5.2.3 we have that

$$\begin{array}{ccc} & TTC & \\ \Pi_f \swarrow & & \searrow \pi_{Fh \circ \pi_f} \\ FD & \xrightarrow{Fh} & FD' \end{array}$$

commutes. However, again by Lemma 5.2.3, we have that

$$\begin{array}{ccc} & TC & \\ \pi_f \swarrow & & \searrow \pi_{Fh \circ f} \\ FD & \xrightarrow{Fh} & FD' \end{array}$$

commutes. So we have

$$Fh \circ \pi_f = \pi_{Fh \circ f} = \pi_{f'}.$$

This gives that

$$\begin{array}{ccc} & C & \\ \Pi_f \swarrow & & \searrow \Pi_{f'} \\ FD & \xrightarrow{Fh} & FD' \end{array}$$

commutes. So the morphisms  $\Pi_f$  form a cone from  $TTC$  to the diagram

$$(C \downarrow F) \xrightarrow{U} \mathcal{D} \xrightarrow{F} \mathcal{C}.$$

□

The endofunctor  $T$  is defined as the limit of this diagram, so the above cone induces a morphism from  $TTC$  to  $TC$ .

**Definition 5.3.6.** We define  $\mu_C$  to be the unique morphism such that the diagram

$$\begin{array}{ccc} TTC & & \\ \mu_C \downarrow & \searrow \Pi_f & \\ TC & \xrightarrow{\pi_f} & FD \end{array}$$

commutes for all morphisms  $C \xrightarrow{f} FD$  in  $\mathcal{C}$ .

Suppose we have an object  $C$  of  $\mathcal{C}$ , thinking of the elements of  $TTC$  and  $TC$  as natural transformations, given an element  $J$  of  $TTC$ , and a morphism  $C \xrightarrow{f} FD$  in  $\mathcal{C}$ , by the above definition, we have

$$\mu_C(J)_D(f) = J_D(\pi_f).$$

**Lemma 5.3.7.** The family of morphisms  $\mu_C$ , for  $C$  an object of  $\mathcal{C}$ , is a natural transformation from  $TT$  to  $T$ .

*Proof.* Suppose we have a morphism  $C \xrightarrow{g} FD$  in  $\mathcal{C}$  and a morphism  $D \xrightarrow{f} D'$  in  $\mathcal{D}$ . Consider the diagram

$$\begin{array}{ccccc} TTC & \xrightarrow{TTg} & TTFD & & \\ \mu_C \downarrow & \Pi_{fg} \searrow & \Pi_f \swarrow & & \downarrow \mu_{FD} \\ & & FD' & & \\ & \pi_{fg} \nearrow & \nwarrow \pi_f & & \\ TC & \xrightarrow{Tg} & TFD & & \end{array}$$

The left and right triangles commute by the definition of  $\mu$ . The lower triangle commutes as a direct application of Lemma 5.2.4 and the upper triangle commutes by two applications of Lemma 5.2.4.

This gives that the diagram

$$\begin{array}{ccccc} TTC & \xrightarrow{TTg} & TTFD & & \\ \mu_C \downarrow & & \downarrow \mu_{FD} & & \\ TC & \xrightarrow{Tg} & TFD & \xrightarrow{\pi_f} & FD' \end{array}$$

commutes. The family of morphisms  $\pi_f$ , for  $f$  a morphism in  $\mathcal{D}$  of the form  $D \xrightarrow{f} D'$ , are the projection morphisms of a limit. So the family of morphisms  $\pi_f \circ Tg \circ \mu_C$  is a cone for this limit.



This gives that there is a unique morphism  $e$  from  $TTC$  to  $TFD$  such that the diagram

$$\begin{array}{ccccc} & & TTC & & \\ & \mu_C \downarrow & \searrow e & & \\ TC & \xrightarrow{Tg} & TFD & \xrightarrow{\pi_f} & FD' \end{array}$$

commutes. However both  $\mu_{FD} \circ TTg$  and  $Tg \circ \mu_C$  satisfy this condition. So, by uniqueness, the diagram

$$\begin{array}{ccc} TTC & \xrightarrow{TTg} & TTFD \\ \mu_C \downarrow & & \downarrow \mu_{FD} \\ TC & \xrightarrow{Tg} & TFD \end{array}$$

commutes, so  $\mu$  is a natural transformation. □

### 5.3.3 The Monad Axioms

We will now see that the endofunctor  $T$  along with the natural transformations  $\eta$  and  $\mu$  defined above form a monad. First we show that  $\mu$  and  $\eta$  satisfy the left and right unit laws for a monad.

**Lemma 5.3.8.** *The natural transformations  $\eta$  and  $\mu$  satisfy the diagram*

$$\begin{array}{ccccc} T & \xrightarrow{T\eta} & TT & \xleftarrow{\eta_T} & T \\ & \searrow 1_T & \downarrow \mu & \swarrow 1_T & \\ & & T & & \end{array}$$

*Proof.* Suppose we have an object  $C$  of  $\mathcal{C}$  and a morphism  $C \xrightarrow{f} FD$  in  $\mathcal{C}$  for some object  $D$  of  $\mathcal{D}$ . Consider the following diagram

$$\begin{array}{ccccc} TC & \xrightarrow{\eta_{TC}} & TTC & \xrightarrow{\mu_C} & TC \\ \pi_f \downarrow & & \downarrow T\pi_f & \searrow \Pi_f & \downarrow \pi_f \\ FD & \xrightarrow{\eta_{FD}} & TFD & \xrightarrow{\pi_D} & FD \\ & \searrow & \text{---} & \swarrow & \\ & & 1_{FD} & & \end{array}$$

The left square commutes as  $\eta$  is a natural transformation and the lower triangle commutes by the definition of  $\eta$ . The upper right triangle commutes by the definition of  $\mu_C$  and the lower right triangle commutes by Lemma 5.2.4 and the definitions of  $\pi_D$  and  $\Pi_f$ . So the outer pentagon commutes that is, the diagram

$$\begin{array}{ccccc} TC & \xrightarrow{\eta_{TC}} & TTC & \xrightarrow{\mu_C} & TC \\ & \searrow \pi_f & & \swarrow \pi_f & \\ & & FD & & \end{array}$$

commutes for all objects  $C$  of  $\mathcal{C}$  and morphisms  $C \xrightarrow{f} FD$  in  $\mathcal{C}$  for  $D$  an object of  $\mathcal{D}$ . The morphisms  $\pi_f$ , for  $f$  a morphism in  $\mathcal{C}$  of the form  $C \xrightarrow{f} FD$ , are the projection morphism for a limit. Which means that the family of morphisms  $\pi_f \circ \mu_C \circ \eta_{TC}$  are a cone for this limit.

So there is a unique morphism  $e$  from  $TC$  to  $TD$  such that  $\pi_f \circ \mu_C \circ \eta_{TC} = \pi_f \circ e$ . However, we then have that

$$\begin{array}{ccc} T & \xrightarrow{\eta_T} & TT \\ & \searrow 1_T & \downarrow \mu \\ & & T \end{array}$$

commutes, so  $\eta$  and  $\mu$  satisfy the left unit law.

Suppose again that we have an object  $C$  of  $\mathcal{C}$  and a morphism  $C \xrightarrow{f} FD$  in  $\mathcal{C}$  for some object  $D$  of  $\mathcal{D}$ . First we observe that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & TC \\ & \searrow f & \downarrow \pi_f \\ & & FD \end{array}$$

commutes by the definition of  $\eta$ , so the diagram

$$\begin{array}{ccc} TC & \xrightarrow{T\eta_C} & TTC \\ & \searrow Tf & \downarrow T\pi_f \\ & & TFD \end{array}$$

also commutes.

Now consider the following diagram

$$\begin{array}{ccccc} TC & & \xrightarrow{T\eta_C} & & TTC \\ & \searrow Tf & & \swarrow T\pi_f & \\ & & TFD & & \\ & \downarrow \pi_B & \swarrow \Pi_f & \downarrow \mu_C & \\ & & FD & \xleftarrow{\pi_f} & TC \end{array}$$

The leftmost and centre triangles commute by Lemma 5.2.3. The upper triangle commutes as it is an example of the previous diagram and the lower right triangle commutes by the definition of  $\mu$ .

So the outer square commutes, that is, for all objects  $C$  of  $\mathcal{C}$  and morphisms  $C \xrightarrow{f} FD$  in  $\mathcal{C}$  with  $D$  an object of  $\mathcal{D}$ , the square

$$\begin{array}{ccccc} TC & \xrightarrow{T\eta_C} & TTC & \xrightarrow{\mu_C} & TC \\ & \searrow \pi_f & & \swarrow \pi_f & \\ & & FD & & \end{array}$$

commutes. So, as the morphisms  $\pi_f$  are the projection morphisms of a limit, using the same argument as above, we have that the diagram

$$\begin{array}{ccc} TT & \xleftarrow{T\eta} & T \\ \mu \downarrow & \swarrow 1_T & \\ T & & \end{array}$$

commutes. So  $\eta$  and  $\mu$  also satisfy the right unit law. □

We now see that  $\mu$  satisfies the associativity law for a monad.

**Lemma 5.3.9.** *The natural transformation  $\mu$  satisfies the diagram*

$$\begin{array}{ccc} TTT & \xrightarrow{T\mu} & TT \\ \mu_T \downarrow & & \downarrow \mu \\ TT & \xrightarrow{\mu} & T \end{array}$$

*Proof.* Suppose we have an object  $C$  of  $\mathcal{C}$  and some morphism  $C \xrightarrow{f} FD$  in  $\mathcal{C}$  for  $D$  an object of  $\mathcal{D}$ . Consider the diagram

$$\begin{array}{ccc} TTTC & \xrightarrow{T\mu_C} & TTC \\ \pi_{\Pi_f} \searrow & & \swarrow T\pi_f \\ & TFD & \\ \pi_{\Pi_f} \searrow & \downarrow \pi_1 \downarrow F & \swarrow \Pi_f \\ & FD & \end{array}$$

The upper triangle commutes as it is just  $T$  applied to the definition of  $\mu$  and the left and right triangles commute by Lemma 5.2.4, so the outer triangle commutes.

Now consider the diagram

$$\begin{array}{ccc} TTTC & \xrightarrow{T\mu_C} & TTC \\ \mu_{TC} \downarrow & \searrow \pi_{\Pi_f} & \downarrow \Pi_f \\ TTC & \xrightarrow{\Pi_f} & D \end{array}$$

The upper triangle is an example of the previous diagram so it commutes and the lower triangle commutes by the definition of  $\mu$  so the outer square commutes.

Finally, consider the diagram

$$\begin{array}{ccc}
TTTC & \xrightarrow{T\mu_C} & TTC \\
\mu_{TC} \downarrow & & \downarrow \mu_C \\
TTC & \xrightarrow{\mu_C} & TC \\
& \searrow \Pi_f & \downarrow \pi_f \\
& & D
\end{array}$$

The outer square is an example of the previous diagram so it commutes and the upper right and lower left triangles commute by the definition of  $\mu$ . So the outer square of the above diagram commutes. So for all objects  $C$  of  $\mathcal{C}$  and morphisms  $C \xrightarrow{f} FD$  in  $\mathcal{C}$  with  $D$  an object of  $\mathcal{D}$ , we have that

$$\begin{array}{ccc}
TTTC & \xrightarrow{T\mu_C} & TTC \\
\mu_{TC} \downarrow & & \downarrow \mu_C \\
TTC & \xrightarrow{\mu_C} & TC \\
& & \downarrow \pi_f \\
& & D
\end{array}$$

commutes.

The family of morphisms  $\pi_f$  for  $f$  a morphism in  $\mathcal{C}$  of the form  $C \xrightarrow{f} FD$ , are the projection morphisms of a limit. So the morphisms  $\pi_f \circ \mu_C \circ \mu_{TC}$  are a cone for this limit. This means that there is a unique morphism  $e$  from  $TTTC$  to  $TC$  such that the diagram

$$\begin{array}{ccc}
TTTC & & \\
\mu_{TC} \downarrow & \searrow e & \\
TTC & \xrightarrow{\mu_C} & TC \\
& & \downarrow \pi_f \\
& & D
\end{array}$$

commutes. However both  $\mu_C \circ T\mu_C$  and  $\mu_C \circ \mu_{TC}$  satisfy this condition, so, by uniqueness, the diagram

$$\begin{array}{ccc}
TTT & \xrightarrow{T\mu} & TT \\
\mu_T \downarrow & & \downarrow \mu \\
TT & \xrightarrow{\mu} & T
\end{array}$$

commutes. That is,  $\mu$  satisfies the multiplication law for a monad. □

Combining the above results we have the following.

**Proposition 5.3.10.** *The endofunctor  $T$  along with the natural transformations  $\eta$  and  $\mu$  are a monad.*

*Proof.* This follows directly from Lemmas 5.3.8 and 5.3.9. □

So, by Proposition 5.3.10,  $T$  along with the morphisms  $\eta$  and  $\mu$  is a monad. We let  **$T$ -Alg** be the category of  $T$ -algebras for this monad and the  $T$ -algebra morphisms between them.

In Chapter 9 we will show that, when  $\mathcal{C}$  is a suitable category of algebras, this category of  $T$ -algebras is isomorphic to the category of algebraically compact algebras.

## Chapter 6

# Ultrafilters and Codensity

Suppose that  $T$  is the endofunctor induced by the inclusion of the subcategory of finite sets into the category **Set**. In [15], Manes proves that for any set  $S$ ,  $TS$  is the set of ultrafilters on  $S$ . In this chapter we will prove an analogous result for algebras.

For the duration of this chapter we will assume, as in Chapter 2, that we have a category  $\mathcal{A}$  of algebras of a theory that contains the theory of groups and a subcategory  $\mathcal{B}$  of finite algebras. The functor  $T$  will denote the endofunctor of the codensity monad induced by the inclusion of  $\mathcal{B}$  into  $\mathcal{A}$ . We will show that, for any algebra  $A$ , the underlying set of the algebra  $TA$  is the set of ultrafilters on  $A$ .

As  $\mathcal{B}$  is a full subcategory of  $\mathcal{A}$ , for convenience we omit the inclusion functor  $F$ . We let  $|-| : \mathcal{A} \rightarrow \mathbf{Set}$  be the forgetful functor that sends an algebra to its underlying set.

### 6.1 Constructing Elements of the Codensity Monad

Suppose we have an algebra  $A$ . Given an ultrafilter  $\mathcal{U}$  on  $A$ , we wish to construct an element of  $TA$ . In the following definition, given an ultrafilter on  $A$ , we construct a corresponding family of morphisms in **Set**. Then, in Lemma 6.1.2 and Proposition 6.1.3 we show that this family of morphisms gives an element of  $TA$ .

**Definition 6.1.1.** *Given an ultrafilter  $\mathcal{U}$  on an algebra  $A$  we define for each object  $B$  of  $\mathcal{B}$  a function  $I_{\mathcal{U},B} : \mathcal{A}(A,B) \rightarrow |B|$  by  $I_{\mathcal{U},B}(f) = \int f d\mathcal{U}$  for each  $f \in \mathcal{A}(A,B)$ .*

We now show that this family of morphisms is natural in  $B$ . We write  $\mathcal{A}(A,-)|_{\mathcal{B}}$  for the functor from  $\mathcal{B}$  to **Set** that results from restricting  $\mathcal{A}(A,-)$  to the subcategory  $\mathcal{B}$ .

**Lemma 6.1.2.** *Given an ultrafilter  $\mathcal{U}$  on an algebra  $A$ , the family of morphisms  $(I_{\mathcal{U},B})_{B \in \mathcal{B}}$  is a natural transformation  $I_{\mathcal{U}} : \mathcal{A}(A,-)|_{\mathcal{B}} \rightarrow |-|$ .*

*Proof.* Given objects  $B$  and  $C$  and a morphism  $B \xrightarrow{h} C$  in  $\mathcal{B}$ , using the above definition and Lemma 3.2.4 we have

$$I_{\mathcal{U},C}(hf) = \int hf d\mathcal{U} = h(\int f d\mathcal{U}) = |h|(\int f d\mathcal{U}) = |h|(I_{\mathcal{U},B}(f))$$

so  $I_{\mathcal{U}}$  is a natural transformation. □

We now see that  $I_{\mathcal{U}}$  is in fact an element of  $TA$ .

**Proposition 6.1.3.** *Given an ultrafilter  $\mathcal{U}$  on an algebra  $A$ , the natural transformation  $I_{\mathcal{U}}$  is an element of  $TA$ .*

*Proof.* As explained in [14] section IX.5, for any pair of functors  $F$  and  $G$  from  $\mathcal{C}$  to  $\mathcal{D}$ , the set of natural transformations from  $F$  to  $G$  is the end,

$$\int_{C \in \mathcal{C}} \text{Hom}(FC, GC).$$

So we have,

$$I_{\mathcal{U}} \in \int_{B \in \mathcal{B}} \text{Hom}(\mathcal{A}(A, B)|_{\mathcal{B}}, |B|).$$

However, the forgetful functor  $| - |$  preserves limits and in **Set** we can replace hom sets by products, so we have

$$\text{Hom}(\mathcal{A}(A, B)|_{\mathcal{B}}, |B|) = [\mathcal{A}(A, B)|_{\mathcal{B}}, |B|] = [[\mathcal{A}(A, B)|_{\mathcal{B}}, B]].$$

From the above we have

$$\int_{B \in \mathcal{B}} \text{Hom}(\mathcal{A}(A, B)|_{\mathcal{B}}, |B|) = \int_{B \in \mathcal{B}} [[\mathcal{A}(A, B), B]]$$

and, again as  $| - |$  preserves limits, this gives,

$$\int_{B \in \mathcal{B}} [[\mathcal{A}(A, B), B]] = | \int_{B \in \mathcal{B}} [\mathcal{A}(A, B), B] | = |TA|$$

so  $I_{\mathcal{U}}$  is an element of  $TA$ . □

## 6.2 Constructing Ultrafilters

Given an algebra  $A$  and an element of  $TA$ , we wish to construct an ultrafilter on  $A$ . In the definition below, given an element of  $TA$ , we construct a family of affine cofinite subsets of  $A$ . Then in Lemma 6.2.2 and Proposition 6.2.3 we show that this family of subsets is an ultrafilter on  $A$ .

**Definition 6.2.1.** *Given an algebra  $A$  and an element  $I$  of  $TA$  we define  $\mathcal{U}_I$  as the collection of affine cofinite subsets of  $A$  given by*

$$\mathcal{U}_I = \{f^{-1}(I(f)) | A \xrightarrow{f} B, B \text{ an object of } \mathcal{B}\}.$$

To show that  $\mathcal{U}_I$  is an ultrafilter, we must first show that it does not contain the empty set.

**Lemma 6.2.2.** *Given an algebra  $A$ , an element  $I$  of  $TA$  and a morphism  $A \xrightarrow{f} B$  with  $B$  an object of  $\mathcal{B}$ , we have  $f^{-1}(I(f)) \neq \emptyset$ .*

*Proof.* For any morphism  $A \xrightarrow{f} B$  with  $B$  an object of  $\mathcal{B}$  consider the image factorization of  $f$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow e \quad \nearrow m & \\ & \text{Im } f & \end{array}$$

As  $\text{Im } f$  is a sub-object of  $B$  which is an object of the pseudovariety  $\mathcal{B}$ ,  $\text{Im } f$  is also an object of  $\mathcal{B}$ . So by naturality of  $I \in TA$  we have

$$I(f) = I(me) = m(I(e))$$

however,  $m$  is the injective inclusion of  $\text{Im } f$  in  $B$ , so  $I(f) \in \text{Im } f$ . That is, we have  $f^{-1}(I(f)) \neq \emptyset$ .  $\square$

We will now show that  $\mathcal{U}_I$  satisfies the definition of an ultrafilter.

**Proposition 6.2.3.** *Given an algebra  $A$  and an element  $I$  of  $TA$  then the collection of sets  $\mathcal{U}_I$  is an ultrafilter on  $A$ .*

*Proof.* From the definition we see that  $\mathcal{U}_I$  is a set of affine cofinite subsets of  $\mathcal{A}$ . Suppose we have a morphism  $A \xrightarrow{f} B$  with  $B$  an object of  $\mathcal{B}$ . We have  $f^{-1}(I(f)) \in \mathcal{U}_I$ , we just have to show that  $I(f)$  is the unique such element of  $B$ .

Suppose we have a  $b \in B$  with  $f^{-1}(b) \in \mathcal{U}_I$ , we must then have a morphism  $A \xrightarrow{g} C$  with  $C$  finite such that  $f^{-1}(b) = g^{-1}(I(g))$ . Consider the intersection of  $f^{-1}(I(f))$  and  $f^{-1}(b)$ , we will have

$$f^{-1}(I(f)) \cap f^{-1}(b) = f^{-1}(I(f)) \cap g^{-1}(I(g)) = \langle f, g \rangle^{-1}(\langle I(f), I(g) \rangle).$$

Now let  $p$  and  $q$  be the projection morphisms from the product  $B \times C$  as below

$$\begin{array}{ccccc} & & A & & \\ & \swarrow f & \downarrow \langle f, g \rangle & \searrow g & \\ B & \xleftarrow{p} & B \times C & \xrightarrow{q} & C \end{array}$$

As  $\mathcal{B}$  is closed under finite products  $B \times C$  will also be in  $\mathcal{B}$ . So we have

$$p(I(\langle f, g \rangle)) = I(p(\langle f, g \rangle)) = I(f)$$

and similarly  $q(I(\langle f, g \rangle)) = I(g)$  which gives

$$I(\langle f, g \rangle) = \langle I(f), I(g) \rangle$$

So we now have

$$f^{-1}(I(f)) \cap f^{-1}(b) = \langle f, g \rangle^{-1}(I(\langle f, g \rangle)) \in \mathcal{U}_I$$

however,  $\emptyset \notin \mathcal{U}_I$ , so  $f^{-1}(I(f)) \cap f^{-1}(b) \neq \emptyset$  so  $b = I(f)$ . That is,  $I(f)$  is the unique element of  $B$  with  $f^{-1}(I(f)) \in \mathcal{U}_I$ .



Now suppose  $S$  is an affine cofinite subset of  $B$  with  $f^{-1}(S) \in \mathcal{U}$ , we will show that  $I(f) \in S$ . As  $S$  is affine cofinite we must have a morphism  $B \xrightarrow{h} D$  with  $D$  an object of  $\mathcal{B}$  and a  $d \in D$  with  $S = h^{-1}(d)$ .

We have

$$(hf)^{-1}(d) = f^{-1}(h^{-1}(d)) = f^{-1}(S) \in \mathcal{U}_I.$$

However, we have seen that  $I(hf)$  is the unique point in  $D$  with this property so we have  $I(hf) = d$ . However this gives

$$h(I(f)) = I(hf) = d$$

we then have

$$I(f) \in h^{-1}(d) = S.$$

So  $\mathcal{U}_I$  is an ultrafilter on  $A$ .

□

### 6.3 The Bijection

In the previous section we saw that, given an algebra  $A$  and an ultrafilter on  $A$ , we can construct an element of  $TA$ . We also saw that given an element of  $TA$  we can construct an ultrafilter on  $A$ . We will now see that these constructions are mutually inverse.

**Proposition 6.3.1.** *Given an algebra  $A$  and an element  $I$  of  $TA$ , we have  $I = I_{\mathcal{U}_I}$ .*

*Proof.* Suppose we have a morphism  $A \xrightarrow{f} B$  in  $\mathcal{A}$  with  $B$  an object of  $\mathcal{B}$ . We have  $f^{-1}(I(f)) \in \mathcal{U}_I$  by the definition of  $\mathcal{U}_I$ , so, by uniqueness, we have  $\int f d\mathcal{U}_I = I(f)$ . This gives

$$I_{\mathcal{U}_I}(f) = \int f d\mathcal{U}_I = I(f)$$

So we have  $I = I_{\mathcal{U}_I}$ .

□

**Proposition 6.3.2.** *Given an algebra  $A$  and an ultrafilter  $\mathcal{U}$  on  $A$ , we have  $\mathcal{U} = \mathcal{U}_{I_{\mathcal{U}}}$ .*

*Proof.* Suppose  $S$  is an affine cofinite subset of  $A$  with  $S \in \mathcal{U}_{I_{\mathcal{U}}}$ . Then we have  $S = f^{-1}(I_{\mathcal{U}}(f))$  for some morphism  $A \xrightarrow{f} B$  with  $B$  finite. Now by definition of  $I_{\mathcal{U}}$  we have

$$I_{\mathcal{U}}(f) = \int f d\mathcal{U}$$

but  $\int f d\mathcal{U}$  is the unique element of  $B$  with  $f^{-1}(\int f d\mathcal{U}) \in \mathcal{U}$  so we have

$$S = f^{-1}(I_{\mathcal{U}}(f)) = f^{-1}(\int f d\mathcal{U}) \in \mathcal{U}$$

which gives,  $\mathcal{U}_{I_{\mathcal{U}}} \subset \mathcal{U}$ .

Suppose now that  $T$  is an affine cofinite subset of  $A$  with  $T \in \mathcal{U}$ . Then, for some  $A \xrightarrow{k} D$  with  $D$  finite and some  $d$  in  $D$ , we have  $T = k^{-1}(d)$ . Now we have  $k^{-1}(d) \in \mathcal{U}$ , so, as  $\mathcal{U}$  is an ultrafilter, we have

$$I_{\mathcal{U}}(k) = \int k d\mathcal{U} = d$$

but this means,  $T = k^{-1}(I_{\mathcal{U}}(k))$  so  $T \in \mathcal{U}_{I_{\mathcal{U}}}$ , which gives,  $\mathcal{U} \subset \mathcal{U}_{I_{\mathcal{U}}}$ , and so  $\mathcal{U} = \mathcal{U}_{I_{\mathcal{U}}}$ .

□

We now combine the above results.

**Theorem 6.3.3.** *Given an algebra  $A$ , there is a bijection between the sets  $|TA|$  and  $\mathfrak{U}(A)$ .*

*Proof.* This follows directly from Propositions 6.3.1 and 6.3.2.  $\square$

## 6.4 The Natural Isomorphism

We have shown that, given an algebra  $A$ , there is a bijection between the sets  $|TA|$  and  $\mathfrak{U}(A)$ . In the following lemma we see that this is in fact a natural isomorphism.

**Theorem 6.4.1.** *The functor  $|T(-)|$  is naturally isomorphic to the functor  $\mathfrak{U}$ .*

*Proof.* In Lemma 6.3.3 we saw that, given an algebra  $A$ , there is a bijection between  $|TA|$  and  $\mathfrak{U}(A)$ . We let  $|TA| \xrightarrow{\sigma_A} \mathfrak{U}(A)$  be this bijection. We will show that  $\sigma_A$  is natural in  $A$ .

Given a morphism  $A \xrightarrow{f} B$  in  $\mathcal{A}$ , consider the square

$$\begin{array}{ccc} |TA| & \xrightarrow{\sigma_A} & \mathfrak{U}(A) \\ |Tf| \downarrow & & \downarrow \mathfrak{U}(f) \\ |TB| & \xrightarrow{\sigma_B} & \mathfrak{U}(B) \end{array}$$

For any  $I \in TA$ , we have

$$\sigma_A(I) = \mathcal{U}_I = \{h^{-1}(I(h)) \mid A \xrightarrow{h} C, C \text{ finite}\}$$

and

$$\mathfrak{U}(f)(\sigma_A(I)) = \mathfrak{U}(f)(\mathcal{U}_I) = f_*(\mathcal{U}_I).$$

However  $f_*(\mathcal{U}_I)$  is defined by,  $X \in f_*(\mathcal{U}_I)$  if and only if  $f^{-1}(X) \in \mathcal{U}_I$ , that is,  $f^{-1}(X) = h^{-1}(I(h))$  for some  $A \xrightarrow{h} C$  with  $C$  an object of  $\mathcal{B}$ .

In the other direction  $Tf(I)$  is the element of  $TB$  given by  $Tf(I)(k) = I(kf)$  for  $B \xrightarrow{k} D$ , with  $D$  an object of  $\mathcal{B}$ . Then  $\sigma_B(Tf(I))$  is the ultrafilter  $\mathcal{U}_{Tf(I)}$  defined by

$$\mathcal{U}_{Tf(I)} = \{k^{-1}(I(kf)) \mid B \xrightarrow{k} D, D \text{ an object of } \mathcal{B}\}.$$

To see that these are equal suppose  $Y$  is an affine cofinite subset of  $B$  with  $Y \in f_*(\mathcal{U}_I)$ , that is  $f^{-1}(Y) \in \mathcal{U}_I$ . As  $Y$  is affine cofinite, there is a morphism  $B \xrightarrow{l} D$  in  $\mathcal{A}$  with  $D$  an object of  $\mathcal{B}$  and a  $d \in D$  such that  $Y = l^{-1}(d)$ .

We have  $f^{-1}(l^{-1}(d)) = f^{-1}(Y) \in \mathcal{U}_I$ , so, as  $\mathcal{U}_I$  is an ultrafilter, we must have

$$d = \int l f d\mathcal{U}_I = I(lf)$$

but this gives  $Y = l^{-1}(I(lf)) \in \mathcal{U}_{Tf(I)}$ . So we have  $f_*(\mathcal{U}_I) \subset \mathcal{U}_{Tf(I)}$ .

Suppose now that  $X$  is an affine cofinite subset of  $B$  with  $X \in \mathcal{U}_{Tf(I)}$ . We then have  $X = k^{-1}(I(kf))$  for some morphism  $B \xrightarrow{k} D$  with  $D$  an object of  $\mathcal{B}$ . However this gives

$$f^{-1}(X) = f^{-1}(k^{-1}(I(kf))) = (kf)^{-1}(I(kf)) \in \mathcal{U}_I$$

so  $X \in f_*(\mathcal{U}_I)$  and so  $\mathcal{U}_{Tf(I)} \subset f_*(\mathcal{U}_I)$ . However this gives  $\mathfrak{U}(f)(\sigma_A(I)) = \sigma_B(Tf(I))$ , so  $\sigma$  is natural in  $A$ .  $\square$

As the functors  $|T(-)|$  and  $\mathfrak{U}$  are naturally isomorphic, for an algebra  $A$  we can endow the set  $\mathfrak{U}(A)$  with the algebra structure of  $TA$  and consider  $\mathfrak{U}$  as an endofunctor of  $\mathcal{A}$ , then by slight abuse of notation,  $\sigma$  is a natural isomorphism between  $T$  and  $\mathfrak{U}$ .

In Chapter 8 we will prove some basic results about this algebra structure on  $\mathfrak{U}(A)$ . However the details of this algebra structure of  $\mathfrak{U}(A)$  remain an open question.

In the following lemma we see how we can now think of limits in terms of the elements of  $TA$ .

**Lemma 6.4.2.** *Given an algebraically compact algebra  $A$ , and an ultrafilter  $\mathcal{U}$  on  $A$ , the limit of  $A$  is the unique point  $\lim(\mathcal{U})$  such that for all morphisms  $A \xrightarrow{f} B$  with  $B$  finite such that  $f^{-1}(I(f))$  is closed, we have*

$$f(\lim(\mathcal{U})) = I(f).$$

*Proof.* This follows from Proposition 4.4.3 and Theorem 6.4.1.  $\square$

## Chapter 7

# Constructing $T$ -Algebras

Suppose we have a compact Hausdorff topology on a set  $S$ . There is a limit morphism from  $\mathfrak{U}(S)$  to  $S$  that sends an ultrafilter to its limit in this topology. In fact, the limit morphisms for compact Hausdorff topologies are exactly the algebras for the ultrafilter monad.

However, this monad is isomorphic to the codensity monad on **Set** induced by the inclusion of finite sets. So limit morphisms for compact Hausdorff topologies are exactly the algebras for the codensity monad induced by the inclusion of finite sets.

We will now prove the analogous result for more general algebras. Given an algebraically compact topology on an algebra  $A$  we will show that the limit morphism from  $\mathfrak{U}(A)$  to  $A$  is a  $T$ -algebra for the codensity monad induced by the inclusion of finite algebras.

### 7.1 The Codensity Monad in Terms of Ultrafilters

For the duration of this section we assume that we have some algebra  $A$ . In the following lemmas we will see how the codensity monad can be understood in terms of ultrafilters. We have explicitly constructed the codensity monad and have seen that, for an algebra  $A$ ,  $\mathfrak{U}(A)$  is isomorphic to  $TA$ . So we can think of the morphisms  $\mu_A$  and  $\eta_A$  of the codensity monad as structure morphisms of the monad with endofunctor  $\mathfrak{U}$ . We will now see how these morphisms can be described explicitly in terms of ultrafilters.

**Lemma 7.1.1.** *For a point  $a$  of  $A$  the element  $ev_a$  of  $TA$  corresponds to the principal ultrafilter  $\mathcal{U}_a$ , under the isomorphism  $|TA| \cong \mathfrak{U}(A)$ .*

*Proof.* The principal ultrafilter  $\mathcal{U}_a$  is the collection of affine cofinite sets of the form  $f^{-1}(f(a))$  for any morphism  $A \xrightarrow{f} B$  with  $B$  finite. However we have  $f^{-1}(f(a)) = f(ev_a(f))$ , so  $\mathcal{U}_a = \mathcal{U}_{ev_a}$ .  $\square$

From this we see that, in terms of ultrafilters,  $\eta_A$  is the morphism that sends a point in  $A$  to the principal ultrafilter at that point.

### 7.2 The Limit Map

We will define the limit morphism for an algebraically compact algebra analogously with the limit morphism for a compact Hausdorff space. First we must show that this in fact gives a morphism in  $\mathcal{A}$ .

**Lemma 7.2.1.** *There is a morphism from  $TA$  to  $A$  that sends an element  $I$  of  $TA$  to the point  $\lim(\mathcal{U}_I)$ .*

*Proof.* We have seen in Lemma 4.4.4 that every ultrafilter on an algebraically compact algebra has a unique limit so this morphism will be well defined. We just have to show that it commutes with the operations of our theory.

Suppose that  $m$  is an  $n$ -ary operation of our theory. So  $m$  is a natural transformation from  $|-|^n$  to  $|-|$ . Given a family  $(I_i)_{1 \leq i \leq n}$  of elements of  $TA$  we wish to show that

$$m_A(\lim(\mathcal{U}_{I_1}), \dots, \lim(\mathcal{U}_{I_n})) = \lim(m_{TA}(I_1, \dots, I_n)).$$

Suppose we have a morphism  $A \xrightarrow{f} B$  with  $B$  finite, such that all the fibres of  $f$  are open in  $A$ . As  $m$  is a natural transformation, we have

$$f(m_A(\lim(\mathcal{U}_{I_1}), \dots, \lim(\mathcal{U}_{I_n}))) = m_B(f(\lim(\mathcal{U}_{I_1})), \dots, f(\lim(\mathcal{U}_{I_n}))).$$

By Lemma 6.4.2, we then have

$$m_B(f(\lim(\mathcal{U}_{I_1})), \dots, f(\lim(\mathcal{U}_{I_n}))) = m_B(I_1(f), \dots, I_n(f))$$

Again, as  $m$  is natural we have

$$m_B(I_1(f), \dots, I_n(f)) = m_{TA}(I_1, \dots, I_n)(f)$$

Putting this all together gives

$$f(m_A(\lim(\mathcal{U}_{I_1}), \dots, \lim(\mathcal{U}_{I_n}))) = m_{TA}(I_1, \dots, I_n)(f)$$

for all  $A \xrightarrow{f} B$  with open fibres and  $B$  finite.

So, again by Lemma 6.4.2, we have

$$m_A(\lim(\mathcal{U}_{I_1}), \dots, \lim(\mathcal{U}_{I_n})) = \lim(m_{TA}(I_1, \dots, I_n))$$

so our limit morphism commutes with the  $n$ -ary operations of our algebra, so it is a morphism in  $\mathcal{A}$ . □

We can now make the following definition.

**Definition 7.2.2.** *For an algebraically compact algebra  $A$  we define the limit morphism,*

$$\mathfrak{A}(A) \xrightarrow{\lim_A} A$$

*as the morphism that sends an ultrafilter on  $A$  to its limit.*

## 7.3 The Limit Map is a $T$ -algebra

For the rest of this chapter we assume that the algebra  $A$  is algebraically compact. We wish to show that the limit morphism of this topology is an algebra for the codensity monad induced

by the inclusion of finite algebras. To do this we must show it satisfies the unit and associative laws.

**Lemma 7.3.1.** *The morphism  $TA \xrightarrow{\lim_A} A$  satisfies*

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & \searrow 1_A & \downarrow \lim_A \\ & & A \end{array}$$

*Proof.* Suppose  $a$  is an element of the algebra  $A$ . As  $\mathcal{U}_a$  is a principal ultrafilter,  $a$  is an element of every set  $S \in \mathcal{U}_a$ , so we certainly have

$$a \in \bigcap_{S \in \mathcal{C}(\mathcal{U}_a)} S.$$

However,  $\lim_A(\mathcal{U}_a)$  is the unique element in this intersection so we have

$$\lim_A(\eta_A(a)) = \lim_A(\mathcal{U}_a) = a,$$

that is,  $\lim_A \circ \eta_A = 1_A$ . □

We will need the following definitions and lemmas to prove the associative law for a  $T$ -algebra.

**Definition 7.3.2.** *If  $S$  is an affine cofinite subset of  $A$ , we define  $\hat{S}$  as the subset of  $TA$  given by*

$$\hat{S} = \{I \in TA \mid S \in \mathcal{U}_I\}.$$

The affine cofinite subsets are the only subsets that we are really interested in. Although it is not clear from the definition that  $\hat{S}$  is affine cofinite, we now see that this is in fact true.

**Lemma 7.3.3.** *If  $S$  is an affine cofinite subset of  $A$  then  $\hat{S}$  is affine cofinite subset of  $TA$ .*

*Proof.* As  $S$  is affine cofinite, we have a morphism  $A \xrightarrow{f} B$  with  $B$  finite such that  $S = f^{-1}(b)$  for some  $b$  in  $B$ . Also suppose we have some ultrafilter  $\mathcal{U}$  on  $A$ . Then  $I_{\mathcal{U}} \in \pi_f^{-1}(b)$  is equivalent to  $I_{\mathcal{U}}(f) = b$  which is itself equivalent to  $f^{-1}(b) \in \mathcal{U}$ . So  $I_{\mathcal{U}} \in \pi_f^{-1}(b)$  if and only if  $S \in \mathcal{U}$ , so we have  $\hat{S} = \pi_f^{-1}(b)$ . □

We now use definition 7.3.2 to characterise the morphism  $\mu_A$  in terms of ultrafilters.

**Lemma 7.3.4.** *Given a  $J \in TTA$  and an affine cofinite subset  $S$  of  $A$ , then we have  $\hat{S} \in \mathcal{U}_J$  if and only if  $S \in \mathcal{U}_{\mu_A(J)}$ .*

*Proof.* Suppose we have a morphism  $A \xrightarrow{f} B$  with  $B$  an object of  $\mathcal{B}$  and an element  $b$  of  $B$  such that  $S = f^{-1}(b)$ . We have  $S \in \mathcal{U}_{\mu_A(J)}$  if and only if  $\mu_A(J)_B(f) = b$ , which by the definition of  $\mu_A$  is equivalent to  $J_B(\pi_f) = b$  which in turn is equivalent to  $\pi_f^{-1}(b) \in \mathcal{U}_J$ , that is  $\hat{S} \in \mathcal{U}_J$ . □

**Definition 7.3.5.** *If  $S$  is an affine cofinite subset of  $A$ , we define  $\bar{S}$  as the subset of  $TA$  given by*

$$\bar{S} = \{I \in TA \mid \lim_A(\mathcal{U}_I) \in S\}.$$

Again, we are only interested in affine cofinite sets so we have the following lemma.

**Lemma 7.3.6.** *If  $S$  is an affine cofinite subset of  $A$  then  $\bar{S}$  is affine cofinite subset of  $TA$ .*

*Proof.* As  $S$  is affine cofinite, we have a morphism  $A \xrightarrow{f} B$  with  $B$  finite such that  $S = f^{-1}(b)$  for some  $b$  in  $B$ . Then

$$\bar{S} = \lim_A^{-1}(f^{-1}(b)) = (f \lim_A)^{-1}(b)$$

and by Lemma 7.2.1  $\lim_A$  is a morphism of  $\mathcal{A}$  so  $\bar{S}$  is affine cofinite.  $\square$

In fact we can use the notation of Definitions 7.3.2 and 7.3.5 to reformulate the notion of openness for affine cofinite sets in an algebraically compact algebra.

**Lemma 7.3.7.** *An affine cofinite subset  $S$  of  $A$ , is open if and only if we have*

$$\bar{S} = \hat{S}.$$

*Proof.* This follows from Proposition 4.4.5.  $\square$

Now we show that our limit morphism satisfies the associative law for an algebra of the codensity monad.

**Lemma 7.3.8.** *The morphism  $TA \xrightarrow{\lim_A} A$  satisfies,*

$$\begin{array}{ccc} TTA & \xrightarrow{T(\lim_A)} & TA \\ \mu_A \downarrow & & \downarrow \lim_A \\ TA & \xrightarrow{\lim_A} & A \end{array}$$

*Proof.* Suppose we have an element  $J$  of  $TTA$  and an open affine cofinite set  $S \subset A$ . If we have  $\lim_A(T(\lim_A)(J)) \in S$  then by Proposition 4.4.3 we have  $S \in \mathcal{U}_{T(\lim_A)(J)}$ . Now, we have  $S \in \mathcal{U}_{T \lim_A(J)}$  if and only if there is an object  $B$  of  $\mathcal{B}$  and a morphism  $A \xrightarrow{f} B$  with

$$S = f^{-1}(T(\lim_A)(J)(f)).$$

However we have

$$T(\lim_A)(J)(f) = J(f \circ \lim_A)$$

so we have

$$\bar{S} = \lim_A^{-1}(S) = \lim_A^{-1}(f^{-1}(J(f \circ \lim_A))) \in \mathcal{U}_J.$$

Putting this all together, we have

$$\lim_A(T(\lim_A)(J)) \in S \Rightarrow \bar{S} \in \mathcal{U}_J.$$

By Lemma 7.3.7, as  $S$  is open, we have  $\bar{S} = \hat{S}$ . However, by Lemma 7.3.4 we have that  $\hat{S} \in \mathcal{U}_J$  if and only if  $S \in \mathcal{U}_{\mu_A(J)}$ . So putting this all together, given an open affine cofinite set  $S \subset A$ , we have

$$\lim_A(T(\lim_A)(J)) \in S \Rightarrow S \in \mathcal{U}_{\mu_A(J)}.$$

However, by Lemma 4.4.2,  $\lim_A(\mathcal{U}_{\mu_A(J)})$  is the unique point with this property, so we must have

$$\lim_A(T(\lim_A)(J)) = \lim_A(\mathcal{U}_{\mu_A(J)}),$$

that is, we have  $\lim_A \circ T(\lim_A) = \lim_A \circ \mu_A$ . □

So combining the above lemmas we have the main result of this chapter.

**Theorem 7.3.9.** *The morphism  $\lim_A$  is an algebra for the condensity monad induced by the inclusion of finite algebras.*

*Proof.* This follows from Lemmas 7.3.1 and 7.3.8. □



## Chapter 8

# Constructing Compact Algebras

Suppose that  $T$  is the endofunctor of the codensity monad induced by the inclusion of finite sets into **Set**. Suppose also that we have an algebra  $TS \xrightarrow{h} S$  for this codensity monad. We can define a topology on  $S$  such that  $h$  is the limit morphism for the ultrafilters on  $S$ . With this topology every ultrafilter  $\mathcal{U}$  will have a unique limit,  $h(\mathcal{U})$ , so by Lemma 4.1.5  $S$  will be compact Hausdorff.

As usual, in this chapter, we suppose that we have some category of algebras  $\mathcal{A}$  and a subcategory of finite algebras  $\mathcal{B}$ . We let  $T$  be the endofunctor of the codensity monad induced by the inclusion of  $\mathcal{B}$  in  $\mathcal{A}$ . We also suppose that we have a  $T$ -algebra  $TA \xrightarrow{h} A$ .

In an analogous fashion to the above, we will define an algebraically compact topology on  $A$  such that  $h$  is the limit morphism for ultrafilters on  $A$ .

### 8.1 Defining a Topology on a Codensity Algebra

In the following lemma we see how limits can be used to characterise the topology of an ordinary compact Hausdorff space.

**Lemma 8.1.1.** *If  $S$  is a compact Hausdorff space then the open subsets of  $S$  are exactly those subsets  $U \subset S$  such that for all ultrafilters  $\mathcal{U}$  on  $S$  we have*

$$\lim \mathcal{U} \in U \Rightarrow U \in \mathcal{U}.$$

*Proof.* By the definition of limit, all the open subsets of  $S$  satisfy this condition. Suppose that  $R \subset A$  is not open, we will construct an ultrafilter  $\mathcal{U}$  on  $A$  with  $\lim \mathcal{U} \in R$  but  $R \notin \mathcal{U}$ .

If, for all  $x \in R$ , we had an open subset  $U_x \subset R$  with  $x \in U_x$ , then  $R$  would be the union of these open  $U_x$  and so would itself be open. So, taking the converse, as  $R$  is not open we must have a  $p \in R$  such that for all open subsets  $V \subset R$  we have  $p \notin V$ .

Consider the set  $\mathcal{F}$  of subsets of  $S$  consisting of  $R^c$  and all the open subsets of  $S$  containing  $p$ . Given any  $P$  and  $Q$  in  $\mathcal{F}$ , if  $P$  and  $Q$  both contain  $p$  then certainly  $P \cap Q \neq \emptyset$ , on the other hand if  $p \in P$ , and  $Q = R^c$  then, as  $P$  is open and  $p \in P$ , we have  $P \not\subset R$  so

$$P \cap Q = P \cap R^c \neq \emptyset.$$

So as  $\mathcal{F}$  has the finite intersection property, by Proposition 3.3.4, it can be extended to an ultrafilter  $\mathcal{U}$ . As  $R^c \in \mathcal{U}$  we must have  $R \notin \mathcal{U}$  and as every open set containing  $p$  is in  $\mathcal{U}$  then we must have

$$\lim \mathcal{U} = p \in R.$$

So the open subsets of  $S$  are exactly those that, for all ultrafilters  $\mathcal{U}$  on  $S$ , satisfy,

$$\lim \mathcal{U} \in U \Rightarrow U \in \mathcal{U}.$$

□

With the help of the following definition and lemma we can rephrase the previous result in familiar terms.

**Definition 8.1.2.** *Given a topological space  $S$  and a subset  $V$  of  $S$ , a point  $p$  of  $S$  is an **adherent point** of  $V$  if every open set containing  $p$  contains at least one point of  $V$ .*

**Lemma 8.1.3.** *Given a topological space  $S$  and a subset  $V$  of  $S$ , a point  $p$  of  $S$  is an **adherent point** of  $V$  if there is some ultrafilter  $\mathcal{U}$  with  $V \in \mathcal{U}$  and  $\lim(\mathcal{U}) = p$ .*

*Proof.* Suppose that  $p$  is an adherent point of  $V$ . Let  $\mathcal{O}_p$  be the family of open sets containing  $p$ . The set  $\mathcal{O}_p \cup \{V\}$  will have the finite intersection property so, by Proposition 3.3.4, we can extend it to an ultrafilter  $\mathcal{U}$ . We then have  $V \in \mathcal{U}$  and  $\lim(\mathcal{U}) = p$ .

In the other direction suppose that we have an ultrafilter  $\mathcal{U}$  with  $V \in \mathcal{U}$  and  $\lim(\mathcal{U}) = p$ . Then, by the definition of a limit, we have that every open set containing  $p$  is in  $\mathcal{U}$ , however as we have  $V \in \mathcal{U}$ , every open set containing  $p$  has non-empty intersection with  $V$ . □

Lemma 8.1.1 says that a subset  $U$  of a compact Hausdorff space  $S$  is open if, for all ultrafilters  $\mathcal{U}$  on  $S$ , we have

$$\lim(\mathcal{U}) \in U \Rightarrow U \in \mathcal{U}$$

taking the converse this is equivalent to

$$U \notin \mathcal{U} \Rightarrow \lim(\mathcal{U}) \notin U$$

however, by taking complements, this is equivalent to

$$U^c \in \mathcal{U} \Rightarrow \lim(\mathcal{U}) \in U^c.$$

So, as the closed sets are exactly the complements of open sets, Lemma 8.1.1 says that a set  $V$  is closed if and only if, for all ultrafilters  $\mathcal{U}$  we have

$$V \in \mathcal{U} \Rightarrow \lim(\mathcal{U}) \in V$$

that is,  $V$  is closed if and only if it contains all its adherent points.

So, in Lemma 8.1.1 we saw how a compact Hausdorff topology could be characterised in terms of the limits of ultrafilters. We will now define a topology on a  $T$ -algebra that generalises this idea.

**Definition 8.1.4.** Given a  $T$ -algebra  $TA \xrightarrow{h} A$ , an affine cofinite subset  $U$  of  $A$  is  **$h$ -open** if for all ultrafilters  $\mathcal{U}$  on  $A$  we have

$$h(I) \in U \Rightarrow U \in \mathcal{U}_I.$$

The empty set  $\emptyset$  is trivially  $h$ -open. We also have that these sets are closed under finite intersections.

**Lemma 8.1.5.** *The  $h$ -open subsets are closed under finite intersections.*

*Proof.* The intersection of no open affine cofinite subsets of  $A$  is  $A$  itself. By Lemma 3.2.16 we have that for all ultrafilters  $\mathcal{U}$ ,  $A \in \mathcal{U}$  so  $\mathcal{A}$  trivially satisfies the openness condition, so the intersection of no open affine cofinite sets is itself open affine cofinite.

Suppose we have two open subsets  $U$  and  $V$  of  $A$  and an  $I \in TA$  with

$$h(I) \in U \cap V.$$

Then we must have  $h(I) \in U$  and  $h(I) \in V$ . However  $U$  and  $V$  are open so we have

$$U \in \mathcal{U}_I \text{ and } V \in \mathcal{U}_I$$

however, by Lemma 3.2.18,  $\mathcal{U}_I$  is closed under intersections, so we have

$$U \cap V \in \mathcal{U}_I.$$

Putting this together we have, for any  $I \in TA$ ,

$$h(I) \in U \cap V \Rightarrow U \cap V \in \mathcal{U}_I.$$

That is,  $U \cap V$  is  $h$ -open and so these  $h$ -open sets are closed under finite intersections.  $\square$

So, as they are closed under finite intersections, these  $h$ -open sets are the basis for a topology.

**Definition 8.1.6.** Given a  $T$ -algebra  $TA \xrightarrow{h} A$ , we define the  **$h$ -topology** as the unique topology which has the  $h$ -open subsets as a basis.

We now see that, in the  $h$ -topology, an affine cofinite subset is open if and only if it is  $h$ -open.

**Lemma 8.1.7.** *Every open affine cofinite set in the  $h$ -topology is basic open.*

*Proof.* Suppose that  $U$  is an open affine cofinite subset of  $A$  in the  $h$ -topology. So, for some indexing set  $\lambda$ , we have a family  $(U_i)_{i \in \lambda}$  of basic open affine cofinite sets with

$$U = \bigcup_{i \in \lambda} U_i.$$

Suppose we have an  $I \in TA$  with  $h(I) \in U$ , then there is some  $j \in \lambda$  with  $h(I) \in U_j$ . However  $U_j$  is basic open so we have  $U_j \in \mathcal{U}_I$ , and as ultrafilters are closed under supersets and  $U$  is

itself affine cofinite, we have  $U \in \mathcal{U}_I$ . Putting this together we have

$$h(I) \in U \Rightarrow U \in \mathcal{U}_I.$$

So  $U$  is basic open. □

So, in this topology the open sets are unions of basic open affine cofinite sets. In Proposition 4.3.9 we saw that affine cofinite sets in an algebraically compact algebra are open if and only if they are closed. We will now show, using the following four lemmas, that in the  $h$ -topology closed affine cofinite sets are open. We will prove the converse in the next section. First we show that the closed affine cofinite sets satisfy the converse condition to that satisfied by open affine cofinite sets.

**Lemma 8.1.8.** *Given an affine cofinite subset  $V$  of  $A$  that is closed in the  $h$ -topology, then for any  $I \in TA$  we have*

$$V \in \mathcal{U}_I \Rightarrow h(I) \in V.$$

*Proof.* By definition,  $V = S^c$  for some open set  $S$ . As  $S$  is open, for some indexing set  $\lambda$ , we have a family  $(S_i)_{i \in \lambda}$  of open affine cofinite sets with  $S = \cup_{i \in \lambda} S_i$ . Suppose for some  $I \in TA$  we have,  $V \in \mathcal{U}_I$ , this is equivalent to

$$(\bigcup_{i \in \lambda} S_i)^c \in \mathcal{U}_I.$$

If, for some  $j \in \lambda$  we had  $S_j \in \mathcal{U}_I$  then we would have

$$\emptyset = S_j \cap (\bigcup_{i \in \lambda} S_i)^c \in \mathcal{U}_I$$

which contradicts that  $\mathcal{U}_I$  is proper, so, for all  $i \in \lambda$ , we have

$$S_i \notin \mathcal{U}_I.$$

As these sets are open, by Lemma 8.1.7, for all  $i \in \lambda$  we have

$$h(I) \notin S_i$$

which gives

$$h(I) \notin \bigcup_{i \in \lambda} S_i$$

so we have

$$h(I) \in (\bigcup_{i \in \lambda} S_i)^c = V.$$

Putting all this together gives

$$V \in \mathcal{U}_I \Rightarrow h(I) \in V.$$

□

As our theory contains the theory of groups we can define a multiplication operation on ultrafilters as follows.

**Definition 8.1.9.** *Given an ultrafilter  $\mathcal{U}$  on an algebra  $A$  and a point  $a \in A$ , we define  $a\mathcal{U}$  as the family of affine cofinite sets given by*

$$a \cdot \mathcal{U} = \{a \cdot S \mid S \in \mathcal{U}\}.$$

In the following lemma we see that the set defined above is in fact an ultrafilter.

**Lemma 8.1.10.** *Given an ultrafilter  $\mathcal{U}$  on  $A$ , and a point  $a \in A$ , then  $a \cdot \mathcal{U}$  is also an ultrafilter on  $A$ .*

*Proof.* Suppose we have a morphism  $A \xrightarrow{f} B$  in  $\mathcal{A}$  with  $B$  in  $\mathcal{B}$ . As  $\mathcal{U}$  is an ultrafilter there is a  $b$  in  $B$  such that  $f^{-1}(b) \in \mathcal{U}$ . We then have

$$a \cdot f^{-1}(b) \in a \cdot \mathcal{U}$$

which, by Lemma 2.2.5, gives

$$f^{-1}(f(a) \cdot b) \in a \cdot \mathcal{U}.$$

We'll now show that  $f(a) \cdot b$  is the unique such element of  $B$ . Suppose for some  $c \in B$  we have

$$f^{-1}(c) \in a \cdot \mathcal{U}$$

then we have

$$a^{-1} \cdot f^{-1}(c) \in \mathcal{U}$$

which, again by Lemma 2.2.5, gives

$$f^{-1}(f(a)^{-1} \cdot c) \in \mathcal{U}.$$

As  $\mathcal{U}$  is an ultrafilter this gives  $f(a) \cdot b = c$ .

Suppose we have some affine cofinite subset  $S$  of  $B$ , with  $f^{-1}(S) \in a \cdot \mathcal{U}$ . Then we have

$$a^{-1} \cdot f^{-1}(S) \in \mathcal{U}$$

which, by Lemma 2.2.5, gives

$$f^{-1}(f(a)^{-1} \cdot S) \in \mathcal{U}.$$

However, as  $\mathcal{U}$  is an ultrafilter, this gives

$$b \in f(a)^{-1} \cdot S$$

that is

$$f(a) \cdot b \in S.$$

So  $a \cdot \mathcal{U}$  is an ultrafilter. □

We now see that this multiplication is in fact compatible with the multiplication on  $TA$ .

**Lemma 8.1.11.** *For an object  $A$  of  $\mathcal{A}$ , an element  $a \in A$  and an ultrafilter  $\mathcal{U}$  on  $A$  we have*

$$ev_a \cdot I_{\mathcal{U}} = I_{a \cdot \mathcal{U}}.$$

*Proof.* Suppose we have a morphism  $A \xrightarrow{f} B$  in  $\mathcal{A}$  with  $B$  an object of  $\mathcal{B}$ . We have that  $f^{-1}(I_{\mathcal{U}}(f))$  is in  $\mathcal{U}$ , so  $a \cdot f^{-1}(I_{\mathcal{U}}(f))$  is in  $a \cdot \mathcal{U}$ . However, by Lemma 2.2.5, we have

$$a \cdot f^{-1}(I_{\mathcal{U}}(f)) = f^{-1}(f(a) \cdot I_{\mathcal{U}}(f)) = f^{-1}((ev_a \cdot I_{\mathcal{U}})(f)).$$

So  $(ev_a \cdot I_{\mathcal{U}})(f)$  is the unique point in  $B$  such that  $f^{-1}((ev_a \cdot I_{\mathcal{U}})(f)) \in a \cdot \mathcal{U}$ , that is

$$ev_a \cdot I_{\mathcal{U}} = I_{a \cdot \mathcal{U}}.$$

□

We use the above result to show that, in the  $h$ -topology, the open affine cofinite subsets are those that satisfy the converse of the condition in Lemma 8.1.1. We of course are still relying on the fact that our theory contains the theory of groups.

**Lemma 8.1.12.** *Suppose we have an algebra  $A$  and an affine cofinite subset  $S$  of  $A$  such that, for all  $I \in TA$ , we have*

$$S \in \mathcal{U}_I \Rightarrow h(I) \in S$$

*then  $S$  is basic open.*

*Proof.* Suppose we have an  $I \in TA$  with  $h(I) \in S$ . By the definition of an ultrafilter, for some  $a \in A$  we have  $a \cdot S \in \mathcal{U}_I$ , this then gives

$$S = a^{-1} \cdot a \cdot S \in a^{-1} \cdot \mathcal{U}_I.$$

By hypothesis we then have  $h(I_{a^{-1} \cdot \mathcal{U}_I}) \in S$ .

By Lemma 8.1.11 and the unit law for  $T$ -algebras we have

$$a^{-1} \cdot h(I) = h(ev_{a^{-1}}) \cdot h(I) = h(ev_{a^{-1}} \cdot I) = h(I_{a^{-1} \cdot \mathcal{U}_I}).$$

So we have

$$h(I) = a \cdot h(I_{a^{-1} \cdot \mathcal{U}_I}) \in a \cdot S.$$

However, we now have

$$h(I) \in a \cdot S \cap S$$

which gives

$$S = a \cdot S \in \mathcal{U}_I$$

putting this together we have

$$h(I) \in S \Rightarrow S \in \mathcal{U}_I,$$

so  $S$  is basic open. □

Combining the above lemmas we have the following result.

**Lemma 8.1.13.** *Given a  $T$ -algebra,  $TA \xrightarrow{h} A$ , if an affine cofinite subset of  $A$  is closed in the  $h$ -topology then it is open.*

*Proof.* Suppose we have an affine cofinite subset  $V$  of  $A$  that is closed in the  $h$ -topology. By Lemma 8.1.8, for all  $I \in TA$ , we have

$$V \in \mathcal{U}_I \Rightarrow h(I) \in V.$$

By Lemma 8.1.12 we then have that  $V$  is open. □

## 8.2 Properties of the $h$ -Topology

In this section we will prove that the  $h$ -topology on  $A$  is algebraically compact. We saw in Lemma 8.1.7 that the open affine cofinite sets form a basis for the  $h$ -topology so the first condition of algebraic compactness is satisfied. So we have to show that  $A$  is a topological algebra, the  $h$ -topology is Hausdorff and that families of closed affine cofinite subsets of  $A$  with the finite intersection property have non-empty intersection.

### 8.2.1 Algebraic Operations are Continuous

To show that  $A$ , with the  $h$ -topology, is a topological algebra we must show that the operations of the algebra are continuous with respect to the  $h$ -topology. We rely repeatedly on the assumption that our theory contains the theory of groups and so do not mention it explicitly. First we will show that the group operation is continuous (in a limited sense).

**Lemma 8.2.1.** *Given an element  $a \in A$ , the function  $a \cdot -$  is continuous with respect to the  $h$ -topology.*

*Proof.* Let  $S$  be a basic open subset of  $A$ . As  $S$  is affine cofinite we have a morphism  $A \xrightarrow{f} B$  with  $B$  an object of  $\mathcal{B}$  and a  $b \in B$  with  $f^{-1}(b) = S$ . By Lemma 2.2.5 we have

$$a^{-1} \cdot (S) = a^{-1} \cdot (f^{-1}(b)) = f^{-1}(a^{-1} \cdot b)$$

so,  $a^{-1} \cdot (S)$  is an affine cofinite subset of  $A$ . Suppose we have some ultrafilter  $\mathcal{U}$  on  $A$ , with  $h(I_{\mathcal{U}}) \in a^{-1}(S)$ , this gives  $a \cdot h(I_{\mathcal{U}}) \in S$ .

However, by Lemma 8.1.11 and the unit law of  $T$ -algebras, we have

$$a \cdot h(I_{\mathcal{U}}) = h(ev_a) \cdot h(I_{\mathcal{U}}) = h(ev_a \cdot I_{\mathcal{U}}) = h(I_{a \cdot \mathcal{U}}).$$

so we have  $h(I_{a \cdot \mathcal{U}}) \in S$ . As  $S$  is open, this gives  $S \in a \cdot \mathcal{U}$ , so we have  $a^{-1}(S) \in \mathcal{U}$ . Putting this together we have

$$h(I_{\mathcal{U}}) \in a^{-1}(S) \Rightarrow a^{-1}(S) \in \mathcal{U}$$

so  $a^{-1} \cdot S$  is open, that is,  $a \cdot -$  is continuous. □

In the following lemma we show that if one fibre of a morphism is open in the  $h$ -topology, then all the fibres are open. We will use this result to prove that the operations of the theory are continuous with respect to the  $h$ -topology.

**Lemma 8.2.2.** *Given a morphism  $A \xrightarrow{f} B$  with  $B$  an object of  $\mathcal{B}$ , if for some  $b \in B$ ,  $f^{-1}(b)$  is non-empty and open, then for all  $b' \in B$ ,  $f^{-1}(b')$  is open.*

*Proof.* Let  $a$  be an element of  $f^{-1}(b)$  and  $b'$  be an element of  $B$ . If  $f^{-1}(b')$  is empty then it is open, so we assume that  $f^{-1}(b')$  is non-empty. Let  $a'$  be a point in  $f^{-1}(b')$ , we have

$$f^{-1}(b') = f^{-1}(f(a')) = f^{-1}(f(a') \cdot f(a^{-1}) \cdot b) = a' \cdot a^{-1} \cdot f^{-1}(b).$$

However, by Lemma 8.2.1,  $a' \cdot a \cdot f^{-1}(b)$  is open so  $f^{-1}(b')$  is open.  $\square$

With the previous two lemmas we can show that all the operations of our theory are continuous.

**Proposition 8.2.3.** *If  $\mu$  is an  $n$ -ary operation of our theory, then it is continuous with respect to the  $h$ -topology.*

*Proof.* Suppose that  $S$  is a basic open subset of  $A$ . As  $S$  is affine cofinite there is a morphism  $A \xrightarrow{f} B$ , with  $B$  an object of  $\mathcal{B}$ , and an element  $b \in B$  such that  $S = f^{-1}(b)$ . The  $n$ -ary product morphism induced by  $f$  is denoted by  $A^n \xrightarrow{f^n} B^n$ . As  $\mu$  is a natural transformation the diagram

$$\begin{array}{ccc} |A| \times \dots \times |A| & \xrightarrow{\mu_A} & |A| \\ |f|^n \downarrow & & \downarrow |f| \\ |B| \times \dots \times |B| & \xrightarrow{\mu_B} & |B| \end{array}$$

commutes, which, in particular, gives

$$\mu_A^{-1}S = \mu_A^{-1}(f^{-1}(b)) = (f^n)^{-1}(\mu_B^{-1}(b)).$$

If we let  $C = \mu_B^{-1}(b)$  we also have

$$(f^n)^{-1}(\mu_B^{-1}(b)) = \bigcup_{(b_1, \dots, b_n) \in C} (f^n)^{-1}(b_1, \dots, b_n) = \bigcup_{(b_1, \dots, b_n) \in C} f^{-1}(b_1) \times \dots \times f^{-1}(b_n).$$

However, by Lemma 8.2.2 as  $S$  is open, all fibres of  $f$  are open, and, as unions and products of open sets are open, so the above set is open. This gives  $\mu_a^{-1}(S)$  is open, so  $\mu$  is continuous.  $\square$

We can also use continuity of the group operation to prove the following lemma.

**Lemma 8.2.4.** *The basic open sets of the  $h$ -topology are closed.*

*Proof.* Suppose that  $S$  is a basic open subset of  $A$ . If  $S$  is the empty set then it is closed, so assume  $S \neq \emptyset$ . By Lemma 8.2.1 all the cosets of  $S$  will also be open. If we let  $R$  be the union of the non-trivial cosets of  $S$ , then  $R$  will be a union of open sets so will itself be open. However we have

$$S = A \setminus R$$

so  $S$  is closed.  $\square$

In Proposition 4.3.9 we saw that in an algebraically compact algebra an affine cofinite set is open if and only if it is closed. By combining the above lemmas we have the same result for the  $h$ -topology.



**Proposition 8.2.5.** *For an affine cofinite subset  $S$  of an algebra  $A$ , the following are equivalent*

1. *For all  $I \in TA$ , we have  $S \in \mathcal{U}_I \Rightarrow h(I) \in S$*
2.  *$S$  is basic open*
3.  *$S$  is open*
4.  *$S$  is closed*

*Proof.* If  $S$  is basic open it is certainly open, and by 8.1.7 we have the converse of this statement. So  $S$  is open if and only if it is basic open. The rest follows from Lemmas 8.2.4, 8.1.8 and 8.1.12.  $\square$

Indeed, we can strengthen the condition of Definition 8.1.6 with the following corollary.

**Corollary 8.2.6.** *An affine cofinite subset  $S$  of an algebra  $A$  is open if and only if, for all ultrafilters  $\mathcal{U}$  on  $A$ , it satisfies*

$$S \in \mathcal{U} \Leftrightarrow h(I_{\mathcal{U}}).$$

We can equivalently formulate this with the notation of Definitions 7.3.2 and 7.3.5 as below.

**Corollary 8.2.7.** *An affine cofinite subset  $S$  of  $A$  is open in the  $h$ -topology if and only if it satisfies*

$$\hat{S} = \bar{S}.$$

## 8.2.2 Compactness

We now prove the third condition of algebraic compactness. Interestingly we do not rely on the  $T$ -algebra axioms to do this.

**Proposition 8.2.8.** *Every family of closed affine cofinite subsets of  $A$  with the finite intersection property has non-empty intersection.*

*Proof.* Suppose we have an indexing set  $\lambda$  and a family,  $(S_i)_{i \in \lambda}$ , of closed affine cofinite subset of  $A$ . By Proposition 3.3.4 we can extend this family of sets to an ultrafilter  $\mathcal{U}$ . By Theorem 6.4.1 this will correspond to an element  $I_{\mathcal{U}}$  of  $TA$ .

For any  $i \in \lambda$ ,  $S_i$  is closed and  $S_i \in \mathcal{U}$  so by Lemma 8.1.8,  $h(I_{\mathcal{U}}) \in S_i$ , so we have

$$h(I_{\mathcal{U}}) \in \bigcap_{i \in \lambda} S_i$$

that is, the family  $(S_i)_{i \in \lambda}$  has non empty intersection.  $\square$

## 8.2.3 Hausdorffness

We will now show that the  $h$ -topology is Hausdorff. To do this we will first define certain distinguished sets and prove that they are affine cofinite and always closed.

**Definition 8.2.9.** *Given an affine cofinite subset  $S$  of  $A$ , we define  $S^*$  by*

$$S^* = \{h(I_{\mathcal{V}}) | \mathcal{V} \in \hat{S}\}.$$

We could equivalently say that for an affine cofinite subset  $S$  of  $A$ ,  $S^*$  is the image of  $\hat{S}$  under  $h$ . We will use the following lemma in the proof that  $S^*$  is closed.

**Lemma 8.2.10.** *Given  $I \in TA$ , a morphism  $A \xrightarrow{f} A'$  and an affine cofinite subset  $R$  of  $A'$ , we have*

$$R \in \mathcal{U}_{Tf(I)} \Leftrightarrow f^{-1}(R) \in \mathcal{U}_I.$$

*Proof.* As  $R$  is affine cofinite there is some morphism  $A' \xrightarrow{g} B$  with  $B$  finite and a  $b \in B$  such that  $R = g^{-1}(b)$ . We saw in Section 5.2 that we have  $Tf(I)(g) = I(gf)$ . So we have

$$R \in \mathcal{U}_{Tf(I)} \Leftrightarrow Tf(I)(g) = b \Leftrightarrow I(gf) = b.$$

We also have

$$I(gf) = b \Leftrightarrow (gf)^{-1}(I(gf)) = (gf)^{-1}(b) \Leftrightarrow f^{-1}(R) \in \mathcal{U}_I.$$

Where we are relying on the fact that  $(gf)^{-1}(I(gf)) \neq \emptyset$  in the first equivalence above. Putting these equivalences together we get

$$R \in \mathcal{U}_{Tf(I)} \Leftrightarrow f^{-1}(R) \in \mathcal{U}_I.$$

□

We have already used the unit law for  $T$ -algebras twice, in the following proof we rely on the associative law for  $T$ -algebras.

**Lemma 8.2.11.** *Given an affine cofinite set  $S$ , the set  $S^*$  is closed affine cofinite.*

*Proof.* By definition, the morphism  $h$  is split epic so it is surjective. By Lemma 7.3.3, as  $S$  is affine cofinite,  $\hat{S}$  is affine cofinite. So, by Proposition 2.4.6, as  $S^*$  is the image of an affine cofinite set under a surjection it is itself affine cofinite.

To see that  $S^*$  is closed suppose that, for some  $I \in TA$ , we have  $S^* \in \mathcal{U}_I$ . Define  $\mathcal{Q}$  by

$$\mathcal{Q} = \{\bar{R} | R \in \mathcal{U}_I\} \cup \{\hat{S}\}.$$

We now show that  $\mathcal{Q}$  has the finite intersection property. Given  $R$  and  $R'$  in  $\mathcal{U}_I$  we have  $R \cap R' \neq \emptyset$  so  $\bar{R} \cap \bar{R}' \neq \emptyset$ . On the other hand, given  $R \in \mathcal{U}_I$ , as  $S^* \in \mathcal{U}_I$ , we have  $R \cap S^* \neq \emptyset$ , so we have

$$R \cap \{h(I_V) | V \in \hat{S}\} \neq \emptyset$$

so there exists a  $V \in \hat{S}$  with  $h(V) \in R$ , that is  $V \in \bar{R} \cap \hat{S}$ , so  $\bar{R} \cap \hat{S} \neq \emptyset$ . As  $\mathcal{Q}$  has the finite intersection property, by Proposition 3.3.4, there is an ultrafilter  $\mathcal{W}$  containing  $\mathcal{Q}$ .

Suppose we have  $R \in \mathcal{U}_I$  then we have  $h^{-1}(R) = \bar{R} \in \mathcal{W}$  so, by Lemma 8.2.10, we have  $R \in \mathcal{U}_{Th(\mathcal{W})}$ , this gives  $\mathcal{U}_I \subset \mathcal{U}_{Th(\mathcal{W})}$ . As they are both ultrafilters, by Lemma 3.3.3, we have  $\mathcal{U}_I = \mathcal{U}_{Th(\mathcal{W})}$ . So, by the associative law for a  $T$ -algebra, we have

$$h(I) = h(Th(\mathcal{W})) = h(\mu_A(\mathcal{W})).$$

As  $\hat{S} \in \mathcal{W}$ , by Lemma 7.3.4, we have  $S \in \mathcal{U}_{\mu_A(\mathcal{W})}$ , but this then gives  $\mathcal{U}_{\mu_A(\mathcal{W})} \in \hat{S}$ . So we

have  $h(I_{\mathcal{U}}) = h(\mu_A(\mathcal{W}))$  and  $\mu_A(\mathcal{W}) \in \hat{S}$ , which gives

$$h(I) \in \{h(I_{\mathcal{V}}) | \mathcal{V} \in \hat{S}\} = S^*.$$

Putting this all together we have

$$S^* \in \mathcal{U}_I \Rightarrow h(I) \in S^*,$$

so, by Proposition 8.2.5, our set  $S^* = \{h(\mathcal{V}) | \mathcal{V} \in \hat{S}\}$  is closed.  $\square$

In the following lemma we see how we can think about the closure of a set in an ordinary topological space in terms of ultrafilters.

**Lemma 8.2.12.** *In a compact Hausdorff space, a point  $x$  is in the closure of a set  $S$ , if and only if there is an ultrafilter  $\mathcal{U}$  with  $S \in \mathcal{U}$  and  $\lim(\mathcal{U}) = x$ .*

*Proof.* The point  $x$  is in the closure of  $S$  if and only if, for every closed set  $V$  we have

$$S \subset V \Rightarrow x \in V.$$

By considering complements and the contrapositive of the above we have that  $x$  is in the closure of  $S$  if and only if, for every open set  $U$  we have

$$x \in U \Rightarrow U \cap S \neq \emptyset.$$

Consider the set  $\mathcal{O}_x \cup \{S\}$ . This set has the finite intersection property so, by Proposition 3.3.4, we can extend it to an ultrafilter  $\mathcal{U}$ . We have  $S \in \mathcal{U}$  and, as  $\mathcal{O}_x$  is a subset of  $\mathcal{U}$ , we also have  $\lim(\mathcal{U}) = x$ .

In the other direction, suppose we have an ultrafilter  $\mathcal{U}$  with  $S \in \mathcal{U}$  and  $\lim(\mathcal{U}) = x$ . As  $\mathcal{U}$  is an ultrafilter with limit  $x$ , every open set containing  $x$  is in  $\mathcal{U}$ . However, as  $S$  is also in  $\mathcal{U}$ , every open set containing  $x$  has non-empty intersection with  $S$ . So  $x$  is in the closure of  $S$ .  $\square$

Now we extend the above result to the general case.

**Lemma 8.2.13.** *If  $S$  is an affine cofinite subset of  $A$  and there is a point  $y$  such that, for every closed affine cofinite subset  $S'$  with  $S \subset S'$ , we have  $y \in S'$ , then there is an ultrafilter  $\mathcal{U}$  with  $S \in \mathcal{U}$  and  $h(\mathcal{U}) = y$ .*

*Proof.* Suppose we have such an affine cofinite  $S$ . Now for any point  $x \in S$  we have  $S \in \mathcal{U}_x$  which gives  $\mathcal{U}_x \in \hat{S}$ , however this in turn gives

$$\{\mathcal{U}_x\}_{x \in S} \subset \hat{S}.$$

So, we have

$$S = \{h(ev_x) | x \in S\} \subset \{h(I_{\mathcal{V}}) | \mathcal{V} \in \hat{S}\} = S^*.$$

By our assumptions about  $S$  and Lemma 8.2.11 we have  $y \in S^*$ . However, by the definition of  $S^*$ , this means that there is an ultrafilter  $\mathcal{V}$  with  $S \in \mathcal{V}$  such that  $h(I_{\mathcal{V}}) = y$ .  $\square$

We now define a Kolmogorov topological space. We will then use the above results to prove that the  $h$ -topology is Kolmogorov and then show that this is equivalent to Hausdorffness. We rely on both of the  $T$ -algebra axioms in this proof.

**Definition 8.2.14.** A topological space is **Kolmogorov** if, for every pair of distinct points, there is an open set that contains one but not the other.

Equivalently, a space is Kolmogorov if for, every pair of distinct points, there is a closed set that contains one but not the other. This is the condition we use in the following lemma.

**Lemma 8.2.15.** The  $h$ -topology is Kolmogorov.

*Proof.* Suppose we have two points  $x$  and  $y$  at which the Kolmogorov condition fails, we will show that  $x = y$ . So, every closed set that contains one of  $x$  or  $y$  contains the other. Consider the collection of affine cofinite subsets of  $TA$  given by

$$\mathcal{Q} = \begin{cases} \hat{S} & : S \in \mathcal{U}_x \\ \bar{R} & : R \in \mathcal{U}_y \end{cases}$$

We now show that  $\mathcal{Q}$  has the finite intersection property. Given affine cofinite sets  $S$  and  $S'$  with  $S \in \mathcal{U}_x$  and  $S' \in \mathcal{U}_x$  we have  $x \in S \cap S'$ , so we have  $\mathcal{U}_x \in \hat{S} \cap \hat{S}'$  which gives  $\hat{S} \cap \hat{S}' \neq \emptyset$ . Similarly for affine cofinite sets  $R$  and  $R'$  with  $R \in \mathcal{U}_y$  and  $R' \in \mathcal{U}_y$  we have  $\mathcal{U}_y \in \bar{R} \cap \bar{R}'$  which gives  $\bar{R} \cap \bar{R}' \neq \emptyset$ .

Now suppose we have  $S \in \mathcal{U}_x$  and  $R \in \mathcal{U}_y$ . If  $S'$  is a closed affine cofinite set with  $S \subset S'$ , as  $x \in S \subset S'$  and as the Kolmogorov condition fails at these points we must have  $y \in S'$ . So by Lemma 8.2.13, we have that there must exist an ultrafilter  $\mathcal{U}$  with  $S \in \mathcal{U}$  and  $h(I_{\mathcal{U}}) = y \in R$ , that is  $\mathcal{U} \in \hat{S} \cap \bar{R}$ , so  $\hat{S} \cap \bar{R} \neq \emptyset$ .

We have seen that  $\mathcal{Q}$  has the finite intersection property so, by Proposition 3.3.4, there is an ultrafilter  $\mathcal{V}$  containing  $\mathcal{Q}$ . Now, given an affine cofinite subset  $R$  of  $A$ , by Lemma 8.2.10, we have  $R \in \mathcal{U}_{Th(\mathcal{V})}$  if and only if  $\bar{R} \in \mathcal{V}$ . So we have

$$R \in \mathcal{U}_y \Rightarrow \bar{R} \in \mathcal{Q} \Rightarrow \bar{R} \in \mathcal{V} \Leftrightarrow R \in Th(\mathcal{V}),$$

that is,  $\mathcal{U}_y \subset \mathcal{U}_{Th(\mathcal{V})}$ . So, by Lemma 3.3.3,  $\mathcal{U}_y = \mathcal{U}_{Th(\mathcal{V})}$ , this gives,  $ev_y = Th(\mathcal{V})$ .

Similarly, by Lemma 7.3.4, given an affine cofinite subset  $S$  of  $A$ , if we have  $\hat{S} \in \mathcal{V}$  then we have  $S \in \mathcal{U}_{\mu_A(\mathcal{V})}$ . So, we have

$$S \in \mathcal{U}_x \Rightarrow \hat{S} \in \mathcal{Q} \Rightarrow \hat{S} \in \mathcal{V} \Rightarrow S \in \mathcal{U}_{\mu_A(\mathcal{V})}.$$

that is,  $\mathcal{U}_x \subset \mathcal{U}_{\mu_A(\mathcal{V})}$ . Again, by Lemma 3.3.3, we have  $\mathcal{U}_x = \mathcal{U}_{\mu_A(\mathcal{V})}$ , this gives,  $ev_x = \mu_A(\mathcal{V})$ .

Using the above, and the associative and unit laws for a  $T$ -algebra, we have

$$x = h(ev_x) = h(\mu_A(\mathcal{V})) = h(Th(\mathcal{V})) = h(ev_y) = y.$$

Now, let's consider the converse of this. Suppose we have that  $x$  and  $y$  are distinct, then, by the above there must be a closed set that contains one but not the other, so the  $h$ -topology is Kolmogorov.  $\square$

A Hausdorff space is always Kolmogorov. Indeed, as we now see, for any theory containing the theory of groups, being Kolmogorov is equivalent to being Hausdorff.

**Proposition 8.2.16.** *The  $h$ -topology is Hausdorff.*

*Proof.* As our algebra is Kolmogorov, if we have two points,  $x$  and  $y$ , there must be an open affine cofinite set  $S$  separating one from the other. Assume that we have  $x \in S$ .

As  $S$  is affine cofinite, for some morphism  $A \xrightarrow{f} B$  with  $B$  an object of  $\mathcal{B}$ , there is a  $b$  in  $B$  with  $S = f^{-1}(b)$ . We then have  $y \in f^{-1}(f(y))$  and by Lemma 8.2.2  $f^{-1}(f(y))$  is open. We also have  $f(y) \neq b$ , which gives  $f^{-1}(b) \cap f^{-1}(f(y)) = \emptyset$ , so the space is Hausdorff.  $\square$

**Theorem 8.2.17.** *Given an algebra  $TA \xrightarrow{h} A$ , the  $h$ -topology is algebraically compact.*

*Proof.* This follows from Lemma 8.1.7 and Propositions 8.2.3, 8.2.8 and 8.2.16.  $\square$

## 8.2.4 Finite Topological Algebras

In Proposition 4.3.2 we saw that a finite algebra is compact if and only if it is discrete. We now prove the equivalent result for  $T$ -algebras. Indeed, the following is a consequence of Theorem 8.2.17, but we prove it directly here.

**Lemma 8.2.18.** *Given an object  $B$  of  $\mathcal{B}$  there is only one  $T$ -algebra  $TB \xrightarrow{h} B$  and the resulting  $h$ -topology is discrete.*

*Proof.* Let  $TB \xrightarrow{h} B$  be a  $T$ -algebra. As  $B$  is an object of  $\mathcal{B}$  we have, by Lemma 5.3.3, that the unit morphism  $B \xrightarrow{\eta} TB$  is an isomorphism. However  $h$  satisfies

$$\begin{array}{ccc} B & \xrightarrow{\eta} & TB \\ & \searrow 1_B & \downarrow h \\ & & B \end{array}$$

so we have  $h = \eta^{-1}$ .

Suppose  $S$  is an affine cofinite subset of  $B$  and, for some ultrafilter  $\mathcal{U}$  on  $B$ , we have  $h(\mathcal{U}) \in S$ . As  $\eta$  is an isomorphism for some  $b$  in  $B$  we have that  $\mathcal{U} = \mathcal{U}_b$ , so by the unit law for a  $T$ -algebra we have

$$b = h(\mathcal{U}_b) \in S.$$

However we then have  $S \in \mathcal{U}_b$ . Putting this together, for all ultrafilters  $\mathcal{U}$  on  $B$  we have

$$h(\mathcal{U}) \in S \Rightarrow S \in \mathcal{U}$$

so  $S$  is open. Therefore, the  $h$ -topology on  $B$  is discrete.  $\square$

## Chapter 9

# An Isomorphism of Categories

Suppose, as before, that we have some category  $\mathcal{A}$  of algebras with a full subcategory,  $\mathcal{B}$ , of finite algebras. In this chapter we explicitly construct the isomorphism

$$\begin{array}{ccc} & \Theta & \\ \text{AlgComp} & \xrightarrow{\quad} & T\text{-Alg} \\ & \Gamma & \end{array}$$

between the category of algebraically compact algebras and the category of  $T$ -algebras for the codensity monad induced by the inclusion of  $\mathcal{B}$  in  $\mathcal{A}$ .

### 9.1 The Functors $\Gamma$ and $\Theta$

First we will define the functor  $\Gamma : T\text{-Alg} \rightarrow \text{AlgComp}$ . To do this we will need the following lemma.

**Lemma 9.1.1.** *Given a morphism*

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ h \downarrow & & \downarrow k \\ A & \xrightarrow{f} & B \end{array}$$

*of  $T$ -algebras, if  $A$  and  $B$  are given the  $h$ -topology and the  $k$ -topology respectively then the morphism  $A \xrightarrow{f} B$  will be continuous.*

*Proof.* Suppose  $S$  is an affine cofinite subset of  $B$  that is open in the  $k$ -topology. Consider the affine cofinite subset  $f^{-1}(S)$  of  $A$ . Given an  $I \in TA$ , by hypothesis, we have

$$h(I) \in f^{-1}(S) \Leftrightarrow f(h(I)) \in S \Leftrightarrow k(Tf(I)) \in S.$$

However  $S$  is open in the  $k$ -topology so we have  $S \in \mathcal{U}_{Tf(I)}$  and, by Lemma 8.2.10, this is equivalent to  $f^{-1}(S) \in \mathcal{U}_I$ . Putting this together gives

$$h(I) \in f^{-1}(S) \Rightarrow f^{-1}(S) \in \mathcal{U}_I$$

which gives that  $f^{-1}(S)$  is open in the  $h$ -topology. So  $f$  is continuous.  $\square$

Having proved the above lemma we are now able to make the following definition.

**Definition 9.1.2.** We define  $\Gamma$  to be the functor from the category of  $T$ -algebras to the category of algebraically compact algebras that sends a  $T$ -algebra  $TA \xrightarrow{h} A$  to  $A$  with the  $h$ -topology and sends a morphism

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ h \downarrow & & \downarrow k \\ A & \xrightarrow{f} & B \end{array}$$

of  $T$ -algebras to the continuous morphism  $A \xrightarrow{f} B$ .

By Theorem 8.2.17,  $\Gamma$  sends  $T$ -algebras to algebraically compact algebras. Also,  $\Gamma$  is simply “forgetting” the  $T$ -algebra structure of the morphisms in  $\mathbf{T-Alg}$ , so it will respect composition and identities, so it is indeed a functor.

Before we can define the functor  $\Theta : \mathbf{AlgComp} \rightarrow \mathbf{T-Alg}$ , we need the following lemma.

**Lemma 9.1.3.** If  $A \xrightarrow{f} B$  is a continuous morphism between algebraically compact algebras  $A$  and  $B$  then

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \lim_A \downarrow & & \downarrow \lim_B \\ A & \xrightarrow{f} & B \end{array}$$

commutes.

*Proof.* Given  $I \in TA$ , suppose we have an open affine cofinite subset  $R$  of  $B$ . Suppose also that we have  $f(\lim_A(I)) \in R$  this gives  $\lim_A(I) \in f^{-1}(R)$ . However  $f$  is continuous and  $R$  is open,  $f^{-1}(R)$  is also open. So, by Lemma 4.4.2 we have  $f^{-1}(R) \in \mathcal{U}_I$ . By Lemma 8.2.10 this is equivalent to  $R \in \mathcal{U}_{Tf(I)}$ .

Putting all this together we have

$$f(\lim_A(I)) \in R \Rightarrow R \in \mathcal{U}_{Tf(I)}.$$

However, by Lemma 4.4.2,  $\lim_B(Tf(I))$  is the unique point with this property, so

$$f(\lim_A(I)) = \lim_B(Tf(I))$$

that is, the above square commutes.  $\square$

We can now make the following definition.

**Definition 9.1.4.** We define  $\Theta$  to be the functor from the category of algebraically compact algebras to the category of  $T$ -algebras that sends an algebraically compact algebra  $A$  to its limit morphism  $TA \xrightarrow{\lim_A} A$  and sends a continuous morphism  $A \xrightarrow{f} B$  to the morphism of  $T$ -algebras

given by

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \lim_A \downarrow & & \downarrow \lim_B \\ A & \xrightarrow{f} & B \end{array}$$

By 7.3.9,  $\Theta$  sends algebraically compact algebras to  $T$ -algebras. Also, as  $T$  is a functor it respects composition and identities, so  $\Theta$  also respects composition and identities, so it is indeed a functor.

## 9.2 The Isomorphism

We now show that the functors  $\Gamma$  and  $\Theta$  are mutually inverse.

**Theorem 9.2.1.** *The categories **AlgComp** and **T-Alg** are isomorphic.*

*Proof.* Suppose we have a  $T$ -algebra  $TA \xrightarrow{h} A$  and an element  $I$  of  $TA$ . We know that  $\Gamma(h)$  is  $A$  with the  $h$ -topology and that  $\Theta(\Gamma(h))$  is the limit morphism of this topology.

So, we have that  $\Theta(\Gamma(h))(I)$  is the unique point in the intersection of the closed sets of the ultrafilter  $\mathcal{U}_I$ . However, if  $R$  is an affine cofinite subset of  $A$ , closed in the  $h$ -topology with  $R \in \mathcal{U}_I$  then, by Proposition 8.2.5, we have  $h(I) \in R$ . So we have

$$h(I) \in \bigcap_{R \in \mathcal{U}_I} R,$$

this gives  $\Theta(\Gamma(h))(I) = h(I)$ . We see directly from the definitions that the functor  $\Theta \circ \Gamma$  is also the identity on morphisms, so we have

$$\Theta \circ \Gamma = 1_{\mathbf{T-Alg}}.$$

In the other direction, given an algebraically compact algebra  $A$ ,  $\Theta(A)$  is the  $T$ -algebra  $TA \xrightarrow{\Theta(A)} A$  given by the limit morphism of the topology on  $A$  and  $\Gamma(\Theta(A))$  is the algebra  $A$  with the  $\Theta(A)$ -topology.

Suppose that  $S$  is a basic open set in  $A$  and for some  $I \in TA$  we have  $S \in \mathcal{U}_I$ . By Proposition 4.3.9,  $S$  is also closed affine cofinite. However, we have that  $\Theta(A)(I)$  is the unique point in the intersection of the closed sets of  $\mathcal{U}_I$  so we have  $\Theta(A)(I) \in S$ .

Putting this together, for all  $I \in TA$ , we have

$$S \in \mathcal{U}_I \Rightarrow \Theta(A)(I) \in S.$$

So, by Proposition 8.2.5,  $S$  is basic open in the topology induced  $\Theta(A)$ . This gives that the basic open sets of  $A$  are basic open in  $\Gamma(\Theta(A))$ . As a consequence of this, the open sets of  $A$  are open in  $\Gamma(\Theta(A))$ .

Suppose now that  $R$  is an affine cofinite subset of  $A$  that is not open, by Lemma 4.3.9  $R$ , is not closed. Let  $R'$  be the intersection of the closed sets containing  $R$ . This is itself closed and contains  $R$ . So we have that  $R$  is a proper subset of  $R'$ . This means that there is a point  $x \in A$  with  $x \notin R$  but for every closed  $Z$  with  $R \subset Z$  we have  $x \in Z$ .



Let  $\mathcal{O}$  be the family of open affine cofinite sets containing  $x$ . Given  $V \in \mathcal{O}$  we have  $V^c$  is closed and  $x \notin V^c$  so  $R \not\subset V^c$ , that is,  $R \cap V \neq \emptyset$ . So the family  $\mathcal{O} \cup \{R\}$  has the finite intersection property.

By Proposition 3.3.4 we can extend  $\mathcal{O} \cup \{R\}$  to an ultrafilter  $\mathcal{U}$ . We have  $R \in \mathcal{U}$  and by Lemma 4.4.2 we know  $\Theta(A)(I_{\mathcal{U}}) = x$  which is not in  $R$ . So, by Proposition 8.2.5, this means that  $R$  is not closed in  $\Gamma(\Theta(A))$ . However, again by Proposition 8.2.5, the closed affine cofinite sets are exactly the open affine cofinite sets, so  $R$  is not open.

We have seen that the open sets of  $A$  are exactly the open sets of  $\Gamma(\Theta(A))$ , that is, we have

$$\Gamma(\Theta(A)) = A.$$

We see directly from the definitions that the functor  $\Gamma \circ \Theta$  is also the identity on morphisms, so we have

$$\Gamma \circ \Theta = 1_{\mathbf{AlgComp}}$$

So  $\Gamma$  and  $\Theta$  are mutually inverse. This gives that **AlgComp** is isomorphic to **T-Alg**. □

# Chapter 10

## Examples

In this chapter we show that two well known “compact” topological algebraic structures, profinite groups and profinite rings are in fact examples of algebraically compact algebras.

### 10.1 Groups

Profinite groups originally arose from Galois theory. They were introduced by Serre in [19]. Good references are [18] and [24]. Suppose we have a field  $k$  and  $f$  is an extension of  $k$ . If  $f/k$  is a galois extension then the Galois group is the group of automorphisms of  $f/k$ . This group,  $\text{Aut}(f/k)$ , can be given a topology known as the Krull topology. With this topology the galois groups are profinite. Indeed, in [23], Waterhouse showed that every profinite group is a galois groups.

Explicitly the profinite groups are the compact Hausdorff totally disconnected groups. However, as we will see, a group satisfies these conditions if and only if it is a limit of finite discrete groups. So we can think of the profinite groups as being “approximately” finite. This is the definition of profinite group that we will use.

**Definition 10.1.1.** A **profinite group** is a topological group that is isomorphic to a limit of finite discrete groups.

Finite groups are, of course, profinite. A non trivial example of a profinite group is the  $p$ -adic integers defined below.

**Definition 10.1.2.** Given a prime  $p$ , the  **$p$ -adic integers** are the infinite sums of the form  $\sum_{i=0}^{\infty} a_i p^i$  for  $a_i \in \{0, 1, \dots, p-1\}$ .

For a given  $p$ , we can define a group structure on the  $p$ -adic integers as follows. Given two  $p$ -adic integers  $\sum_{i=0}^{\infty} a_i p^i$  and  $\sum_{i=0}^{\infty} a'_i p^i$ , we define the sequence  $(a + a')_i$  for  $i \geq 0$  as follows. First, when  $i = 0$ , we let

$$(a + a')_0 = a_0 + a'_0 \bmod p.$$

For  $i > 0$ , if  $a_{i-1} + a'_{i-1} \geq p$  then

$$(a + a')_i = a_i + a'_i + 1 \bmod p$$

otherwise

$$(a + a')_i = a_i + a'_i \bmod p.$$

We can then define addition of  $p$ -adic integers by

$$\sum_{i=0}^{\infty} a_i p^i + \sum_{i=0}^{\infty} a'_i p^i = \sum_{i=0}^{\infty} (a + a')_i p^i.$$

We denote the group of  $p$ -adic integers for a given prime  $p$  by  $\mathbb{Z}_p$ .

Suppose we have some prime  $p$ . Consider the following subcategory of the category of groups, the objects of this category are the groups  $\mathbb{Z}/p^i\mathbb{Z}$  and the morphisms are the canonical inclusions. If we consider this subcategory a diagram in the category of groups, then the  $p$ -adic integers are the limit of this diagram, for the details see [20]. So, as each of these groups are finite, the  $p$ -adic integers are profinite.

Most groups aren't profinite. However, we can define a profinite completion on the category of groups. Given a group  $G$  the profinite completion of  $G$  is a profinite group,  $PG$ , that comes with a universal morphism  $G \xrightarrow{\eta} PG$ . The universal property of this morphism is that, given a profinite group  $Q$  and a morphism  $G \xrightarrow{f} Q$ , there is a unique continuous morphism  $PG \xrightarrow{e} Q$  such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\eta} & PG \\ & \searrow f & \downarrow e \\ & & Q \end{array}$$

commutes.

The profinite completion of the integers is the group

$$\prod_{p \text{ prime}} \mathbb{Z}_p$$

with the morphism that, for each prime  $p$ , sends an integer  $z$  to  $(z \bmod p^i)_{i \in \mathbb{N}}$ .

Explicitly, the profinite completion of a group is the limit of all its finite quotients. In order to construct this limit, we make the following definitions.

**Definition 10.1.3.** Given a group  $G$ , we let  $\mathcal{N}_G$  be the category whose objects are normal subgroups of  $G$  with finite index and whose morphisms are the inclusions of normal subgroups.

**Definition 10.1.4.** Given a group  $G$ , we let  $J_G$  be the functor from  $\mathcal{N}_G$  to the category **Grp** that sends an object  $N$  of  $\mathcal{N}_G$  to the group  $G/N$ , and sends a morphism  $N \subseteq M$  in  $\mathcal{N}_G$  to the canonical morphism from  $G/N$  to  $G/M$ .

Given a group  $G$ , there is a canonical cone from  $G$  to the diagram  $J_G$ , consisting of the canonical projection maps  $G \rightarrow G/N$  for  $N$  an object of  $\mathcal{N}_G$ .

**Definition 10.1.5.** Given a group  $G$ , the **profinite completion**  $PG$  of  $G$  is the limit of the functor  $J_G$  along with the morphism from  $G$  to  $PG$  induced by the canonical cone from  $G$  to  $J_G$ .

We usually refer to the group  $PG$  itself as the profinite completion of  $G$ . Profinite completion is in fact an endofunctor of **Grp**. To see this, suppose we have groups  $G$  and  $G'$  and a group

homomorphism,  $G \xrightarrow{f} G'$ . If  $N$  is an object of  $\mathcal{N}_{G'}$ , then  $f^{-1}(N)$  is an object of  $\mathcal{N}_G$ . So there is a canonical projection morphism  $G \rightarrow G/f^{-1}(N)$  and a canonical morphism  $G/f^{-1}(N) \rightarrow G'/N$ , the composite gives a morphism  $G \rightarrow G'/N$ . This family of morphisms, for  $N$  an object of  $\mathcal{N}_{G'}$ , is a cone from  $PG$  to  $J_{G'}$ . So this induces a unique map from  $PG$  to  $PG'$ . We define this explicitly below.

**Definition 10.1.6.** *Given a morphism  $G \xrightarrow{f} G'$  in  $\mathbf{Grp}$ , we define  $Pf$  as the unique morphism such that square*

$$\begin{array}{ccc} PG & \xrightarrow{Pf} & PG' \\ \downarrow & & \downarrow \\ G/f^{-1}(N) & \longrightarrow & G'/N \end{array}$$

*commutes for all objects  $N$  of  $\mathcal{N}_G$ .*

We omit the proof that  $P$  respects identities and composition.

Consider the subcategory of  $\mathbf{Grp}$  consisting of the finite groups. This is a small category closed under taking products, images and subgroups. The finite groups also satisfy the descending chain condition. So we can take this subcategory to be a subcategory of finite algebras as in Definition 2.3.3. We let  $T$  be the endofunctor of the codensity monad induced by the inclusion of finite groups in  $\mathbf{Grp}$ . We will now see that profinite completion is in fact the endofunctor of this codensity monad.

The following proof is lightly adapted from Theorem 3.1 in [5].

**Lemma 10.1.7.** *Given a group  $G$ ,  $TG$  is isomorphic to  $PG$ .*

*Proof.* We let  $i$  be the canonical inclusion functor of the subcategory of finite groups and  $L$  be the canonical inclusion functor from  $\mathcal{N}_G$  to  $(G \downarrow i)$ . We will show that this functor  $L$  is initial. Given an object  $G \xrightarrow{f} FH$  of  $(G \downarrow i)$ , consider the category  $(L \downarrow i)$ . We will show that this category is non-empty and connected.

As  $H$  is a finite group,  $\ker f$  is a normal subgroup of  $G$  with finite index, that is  $\ker f$  is an object of  $\mathcal{N}_G$ . So there is a commutative diagram of the form

$$\begin{array}{ccc} G & \xrightarrow{p} & i(G/\ker f) \\ & \searrow f & \downarrow q \\ & & iH \end{array}$$

so, the comma category,  $(L \downarrow f)$ , is non-empty.

Now suppose that we have two objects in the comma category  $(L \downarrow f)$  as below

$$\begin{array}{ccc} G & \xrightarrow{p_N} & i(G/N) \\ \downarrow p_M & \searrow f & \downarrow s \\ i(G/M) & \xrightarrow{t} & iH \end{array}$$

Consider the subgroup  $N \cap M$  of  $G$ , this is a normal subgroup of  $G$ , and will also be of finite

index. So we have a commutative diagram

$$\begin{array}{ccccc}
G & & \xrightarrow{p_N} & & i(G/N) \\
& \searrow p_{N \cap M} & & \nearrow & \downarrow s \\
& & i(G/(N \cap M)) & & \\
p_M \downarrow & & \nearrow & & \downarrow \\
i(G/M) & & \xrightarrow{t} & & iH
\end{array}$$

However this gives that the comma category  $(L \downarrow f)$  is connected. So the functor  $L$  is initial so, by Theorem IX.3.1 of [14], we have

$$TG = \lim(iU) \cong \lim(J_G) = PG$$

that is,  $TG$  is the profinite completion of  $G$ . □

We can extend the above result as follows.

**Proposition 10.1.8.** *The functor  $T$  is naturally isomorphic to the functor  $P$ .*

*Proof.* By the previous lemma, given a group  $G$  we have an isomorphism  $\alpha_G : TG \cong PG$ . Suppose we have groups  $G$  and  $G'$  and a group homomorphism  $G \xrightarrow{f} G'$ . For  $N$  an object of  $\mathcal{N}_{G'}$  consider the diagram

$$\begin{array}{ccc}
TG & \xrightarrow{Tf} & TG' \\
\downarrow \alpha_G & & \downarrow \alpha_{G'} \\
PG & \xrightarrow{Pf} & PG' \\
\downarrow & & \searrow \\
G/f^{-1}(N) & \xrightarrow{\quad} & G'/N
\end{array}$$

where all unlabelled morphisms are appropriate projections and canonical morphisms. The leftmost and rightmost triangles commute by the definition of  $\alpha$ , the lower square commutes by the definition of  $Pf$  and the outer square commutes by the definition of  $Tf$ . So by the universal property of the limit  $PG'$  we have that the square

$$\begin{array}{ccc}
TG & \xrightarrow{Tf} & TG' \\
\alpha_G \downarrow & & \downarrow \alpha_{G'} \\
PG & \xrightarrow{Pf} & PG'
\end{array}$$

that is,  $\alpha$  is a natural isomorphism. □

As a corollary of the above we have that, the unit morphism  $1_{\mathbf{Grp}} \xrightarrow{\eta} T$ , is the universal morphism of the profinite completion. We now generalise the above results for other subcategories of  $\mathbf{Grp}$ , the finite  $p$ -groups and the finite abelian groups.

**Definition 10.1.9.** A finite  **$p$ -group** is a group of order  $p^n$  for some prime  $p$  and natural number  $n$ .

**Definition 10.1.10.** A **pro- $p$  group** is a group which is a limit of finite discrete  $p$ -groups.

We can define two generalisations of the profinite completion. First we define two categories of normal subgroups. For fixed  $G$  these will be subcategories of the previous  $\mathcal{N}_G$ .

**Definition 10.1.11.** Given a group  $G$  and a prime  $p$ , we let  $\mathcal{N}_{G,p}$  be the category whose objects are normal subgroups of  $G$  with index  $p^n$  for any  $n \in \mathbb{N}$  and whose morphisms are the inclusions of normal subgroups.

**Definition 10.1.12.** Given a group  $G$ , we let  $\mathcal{N}_{G,Ab}$  be the category whose objects are normal subgroups  $N$  of  $G$  such that  $G/N$  is finite abelian and whose morphisms are the inclusions of normal subgroups.

These categories give diagrams in **Grp**.

**Definition 10.1.13.** Given a group  $G$ , we let  $J_{G,p}$  be the functor from  $\mathcal{N}_{G,p}$  to the category **Grp** that sends an object  $N$  of  $\mathcal{N}_{G,p}$  to the group  $G/N$ , and sends a morphism  $N \subseteq M$  in  $\mathcal{N}_{G,p}$  to the canonical morphism from  $G/N$  to  $G/M$ .

**Definition 10.1.14.** Given a group  $G$ , we let  $J_{G,Ab}$  be the functor from  $\mathcal{N}_{G,Ab}$  to the category **Grp** that sends an object  $N$  of  $\mathcal{N}_{G,Ab}$  to the group  $G/N$ , and sends a morphism  $N \subseteq M$  in  $\mathcal{N}_{G,Ab}$  to the canonical morphism from  $G/N$  to  $G/M$ .

As before, we use these diagrams to define completions of a group as a limit of quotients. Given a group  $G$ , we again have canonical cones from  $G$  to the diagrams  $J_{G,p}$  and  $J_{G,Ab}$ .

**Definition 10.1.15.** Given a group  $G$ , the **pro- $p$  completion**  $P^p G$  of  $G$  is the limit of the functor  $J_{G,p}$  along with the morphism from  $G$  to  $P^p G$  induced by the canonical cone from  $G$  to  $J_{G,p}$ .

**Definition 10.1.16.** Given a group  $G$ , the **abelian profinite completion**  $P^{Ab} G$  of  $G$  is the limit of the functor  $J_{G,Ab}$  along with the morphism from  $G$  to  $P^{Ab} G$  induced by the canonical cone from  $G$  to  $J_{G,Ab}$ .

As a direct generalisation of the previous result we will see that these completions are in fact the endofunctors of the codensity monads induced by the inclusion of the appropriate subcategories of finite algebras.

First the pro- $p$  case. Consider the subcategory of **Grp** consisting of finite  $p$  groups. This is a small subcategory that is closed under products, and, by Lagrange's theorem, it is also closed under images and subgroups. The finite  $p$  groups also satisfy the descending chain condition. So we can take this subcategory to be a subcategory of finite algebras as in Definition 2.3.3. We let  $T^p$  be the endofunctor of the codensity monad induced by the inclusion of finite  $p$  groups in **Grp**. We have the following corollary of Proposition 10.1.8.

**Corollary 10.1.17.** The functor  $T^p$  is naturally isomorphic to the functor  $P^p$ .

*Proof.* The proof of Lemma 10.1.7 relied on the fact that the intersection of two normal subgroups of finite index is also a normal subgroup of finite index. We prove the analogous result for the pro- $p$  case.

Suppose, given a prime  $p$ , we have two normal subgroups  $N$  and  $M$  of a group  $G$  with indexes  $p^n$  and  $p^m$  for  $n, m \in \mathbb{N}$ . The subgroup  $N \cap M$  of  $G$ , is a normal subgroup of  $G$ . We have that  $|G/N| = p^n$  and  $|G/M| = p^m$ , so we have  $|G/N \times G/M| = p^{n+m}$ . However there is a canonical injection from  $G/(N \cap M)$  to  $G/N \times G/M$ , so, by Lagrange's theorem, we have that the order of  $G/(N \cap M)$  divides  $|G/N \times G/M| = p^{n+m}$ . So,  $N \cap M$  is of index  $p^l$  for some  $l \in \mathbb{N}$ .

With the above observation the proofs of Lemma 10.1.7 and Proposition 10.1.8 generalise directly to the pro- $p$  case, so  $T^p G$  is now the pro- $p$  completion of  $G$ .  $\square$

We now consider the finite abelian case. The category of finite abelian groups is a small subcategory of **Grp** that is closed under products, subalgebras and quotients. The finite abelian groups also satisfy the descending chain condition. So we can take this subcategory to be a subcategory of finite algebras as in Definition 2.3.3. We let  $T^{ab}$  be the endofunctor of the codensity monad induced by the inclusion of finite abelian groups.

**Corollary 10.1.18.** *The functor  $T^{Ab}$  is naturally isomorphic to the functor  $P^{Ab}$ .*

*Proof.* This is a direct generalisation of Lemma 10.1.7 and Proposition 10.1.8.  $\square$

We have defined profinite groups as limits of finite groups. However, there is an equivalent topological characterisation of profinite groups that does not involve limits or finiteness. We will prove this equivalence explicitly and use this result to show that the profinite groups are exactly the algebraically compact groups. This result requires the following definition.

**Definition 10.1.19.** *A topological space is **totally disconnected** if the connected components are the singletons.*

We also require a number of lemmas which follow. Lemmas 10.1.20, 10.1.21, 10.1.22, 10.1.23 and Proposition 10.1.24 are lightly adapted from [18], specifically from Lemma 1.1.11, Theorem 1.1.12 and Theorem 2.1.3.

**Lemma 10.1.20.** *In a compact Hausdorff space, the connected component of a point is the intersection of the clopen sets containing it.*

*Proof.* Let  $S$  be a compact Hausdorff space. Suppose we have a point  $s \in S$ , for some indexing set  $\lambda$ , let  $(C_i)_{i \in \lambda}$  be the family of clopen sets containing  $s$ . Consider the intersection

$$C = \bigcap_{i \in \lambda} C_i.$$

Any clopen set containing  $s$ , also contains the connected component of  $s$ , otherwise we could partition the connected component into disjoint non-empty closed sets and it would not be connected. So  $C$  contains the connected component of  $s$ , we will now show the reverse inclusion by proving that  $C$  is connected.

Assume that  $C = V_1 \cup V_2$  with  $V_1$  and  $V_2$  closed and disjoint. As they are closed,  $V_1$  and  $V_2$  are also compact, and so there are open sets  $U_1$  and  $U_2$  with  $V_1 \subset U_1$  and  $V_2 \subset U_2$  and  $U_1 \cap U_2 = \emptyset$ .

Now  $C \subset U_1 \cup U_2$  so we have

$$G \setminus (U_1 \cup U_2) \cap \bigcap_{i \in \lambda} C_i = (G \setminus (U_1 \cup U_2)) \cap C = \emptyset.$$

However  $G \setminus (U_1 \cup U_2)$  is closed, so, by compactness, there is some finite indexing set  $I \subset \lambda$  with

$$(G \setminus (U_1 \cup U_2)) \cap \bigcap_{i \in I} C_i = \emptyset.$$

Now let  $B = \bigcap_{i \in I} C_i$ , this set contains  $s$  and, as  $I$  is finite, it is clopen. We have  $B \subset U_1 \cup U_2$ , so consider the partition of  $B$  given by

$$B = (B \cap U_1) \cup (B \cap U_2).$$

Now  $B \cap U_2$  is open and  $B$  is closed so we have

$$(G \setminus (B \cap U_2)) \cap B = B \cap U_1$$

so  $B \cap U_1$  is closed, so it is clopen. Similarly  $B \cap U_2$  is also clopen. We have  $s \in B$  so,  $s$  must be an element of one of these clopen sets. Suppose we have  $s \in B \cap U_1$ , then we would have

$$C \subset B \cap U_1 \subset U_1.$$

However this gives

$$C \cap V_2 \subset C \cap U_2 \subset U_1 \cap U_2 = \emptyset$$

so, as  $V_2 \subset C$ , we have  $V_2 = \emptyset$ , that is  $C$  is connected. So  $C$  is the connected component of  $s$ .  $\square$

**Lemma 10.1.21.** *A point  $s$  in a compact Hausdorff totally disconnected space has a clopen filter basis  $(C_i)_{i \in \lambda}$  with intersection*

$$\bigcap_{i \in \lambda} C_i = \{s\}.$$

*Proof.* Let  $S$  be a compact Hausdorff totally disconnected space. Let  $s$  be a point in  $S$  and  $U$  an open set containing  $s$ . Given an indexing set  $\lambda$ , let  $(C_i)_{i \in \lambda}$  be the family of clopen sets containing  $s$ .

The space  $S$  is totally disconnected so the connected component of the point  $s$  is the set  $\{s\}$ , and we have  $s \notin S \setminus U$  so, by Lemma 10.1.20, we have

$$\left(\bigcap_{i \in \lambda} C_i\right) \cap (S \setminus U) = \emptyset.$$

However, the set  $S \setminus U$  is closed, so, by compactness, there must be a finite indexing set  $I \subset \lambda$ , with

$$\left(\bigcap_{i \in I} C_i\right) \cap (S \setminus U) = \emptyset.$$

The set  $\bigcap_{i \in I} C_i$  is clopen, it contains  $s$  and we have  $\bigcap_{i \in I} C_i \subset U$ . That is, the clopen sets



$(C_i)_{i \in \lambda}$ , are a filter basis for the point  $\{s\}$  with

$$\{s\} = \bigcap_{i \in \lambda} C_i.$$

□

**Lemma 10.1.22.** *If  $G$  is a compact Hausdorff totally disconnected group, then  $G$  has a neighbourhood basis of  $1_G$  of open subgroups with trivial intersection.*

*Proof.* Let  $G$  be a compact Hausdorff totally disconnected group. From Lemma 10.1.21 we see that  $1_G$  has a neighbourhood basis of clopen sets with trivial intersection. So given a clopen set  $V$  containing  $1_G$  we will construct an open subgroup inside  $V$ .

Let  $F = V^2 \cap (G \setminus V)$ , as  $V$  is open  $G \setminus V$  is closed. However,  $V$  is compact so  $V \times V$  is also compact, and hence, so is its image,  $V^2$ , under the group operation. So, as  $V^2$  is compact it is closed, which means that  $F$  is also closed and hence compact.

Now, given  $x \in V$ , we have that  $G \setminus F$  is an open neighbourhood of  $x$ . The group operation sends  $(x, 1_G)$  to  $x$ , so consider the preimage of  $G \setminus F$  under the group operation, this will give a pair of open sets  $V_x$  and  $S_x$  containing  $x$  and  $1_G$  with  $V_x \cdot S_x \subset G \setminus F$ . Consider the intersections  $V'_x = V_x \cap V$  and  $S'_x = S_x \cap V$ , these will also be open, contain  $x$  and  $1_G$  and we will have  $V'_x, S'_x \subset V$  and  $V'_x \cdot S'_x \subset V_x \cdot S_x \subset G \setminus F$ .

For  $x \in V$ , the sets  $V'_x$  form a cover of  $V$ , so, by compactness of  $V$ , there exists some finite  $n$  and points  $x_i$  with  $1 \leq i \leq n$  such that the  $V'_{x_i}$  also cover  $V$ . Now we define the sets

$$S = \bigcap_{1 \leq i \leq n} S'_{x_i}$$

and

$$W = S \cap S^{-1}.$$

As it is an intersection of finitely many open sets  $S$  itself is open and so is  $W$ .

We have

$$VW \subset \left( \bigcup_{1 \leq i \leq n} V'_{x_i} \right) \cdot S = \left( \bigcup_{1 \leq i \leq n} V'_{x_i} \right) \cdot \left( \bigcap_{1 \leq i \leq n} S_{x_i} \right).$$

However we also have

$$V'_{x_i} \cdot \bigcap_{1 \leq i \leq n} S'_{x_i} \subset V'_{x_i} \cdot S'_{x_i} \subset G \setminus F$$

for  $1 \leq i \leq n$  which gives

$$\left( \bigcup_{1 \leq i \leq n} V'_{x_i} \right) \cdot \left( \bigcap_{1 \leq i \leq n} S_{x_i} \right) = \left( \bigcup_{1 \leq i \leq n} V'_{x_i} \cdot \left( \bigcap_{1 \leq i \leq n} S_{x_i} \right) \right) \subset G \setminus F.$$

Putting this together we have

$$VW \subset G \setminus F$$

that is,  $VW \cap F = \emptyset$ . As  $W \subset V$  we have  $VW \subset V^2$ , which gives  $VW \cap (G \setminus V) = \emptyset$ , that is  $VW \subset V$ . We then have  $VW^n \subset V$  for all  $n \in \mathbb{N}$  so we have  $W^n \subset VW^n \subset V$  for all  $n \in \mathbb{N}$ .

We can also rewrite  $W^n$  as

$$W^n = \bigcup_{w \in W} w \cdot W^{n-1}$$

and so, by induction,  $W^n$  is open.

Consider the union

$$U = \bigcup_{n \in \mathbb{N}} W^n,$$

this will be an open subgroup of  $G$  contained in  $V$ . □

**Corollary 10.1.23.** *If  $G$  is a compact Hausdorff totally disconnected group, then there  $G$  has a neighbourhood basis of  $1_G$  composed of open normal subgroups with trivial intersection.*

*Proof.* By the Lemma 10.1.22  $G$  has a neighbourhood basis of  $1_G$  of open subgroups with trivial intersection. Given such a subgroup  $U$  consider it's core

$$N = \bigcap_{g \in G} (g^{-1}Ug).$$

By the orbit stabiliser theorem, the number of distinct conjugates of  $U$  is equal to the index of the normaliser of  $U$ . However  $U$  is normal and, as it is an open subgroup of a compact group, of finite index.

As there are only finitely many distinct conjugates of  $U$ ,  $N$  is an intersection of finitely many distinct open sets. So  $N$  is an open normal subgroup contained in  $U$ . □

With the proceeding lemmas we can now prove the topological characterisation of profinite groups.

**Proposition 10.1.24.** *A group is profinite if and only if it is compact, Hausdorff and totally disconnected.*

*Proof.* Suppose we have a profinite group  $G$ . The discrete topology on finite groups is compact, Hausdorff and totally disconnected so, as  $G$  is a limit in the category of topological groups, it is also compact, Hausdorff and totally disconnected, see Proposition 1.1.5 of [24].

In the other direction, suppose that  $G$  is a compact, Hausdorff, totally disconnected group. Let  $\mathcal{N}$  be the category with objects the open normal subgroups of  $G$  and morphisms their inclusions. Also let  $J$  be the functor from  $\mathcal{N}$  to **Grp** that sends an object  $N$  of  $\mathcal{N}$  to the group  $G/N$  and sends an inclusion  $N \subset M$  to the canonical morphism between  $G/N$  and  $G/M$ .

For  $N$  an open normal subgroup of  $G$  we let  $G \xrightarrow{p_N} G/N$  be the canonical quotient morphism and  $\lim J \xrightarrow{\pi_N} G/N$  be the canonical projection morphism. As  $N$  is an open normal subgroup and  $G$  is compact it has only finitely many cosets, so the quotient  $G/N$  is a finite discrete group. This gives that  $\lim J$  is a profinite group.

The quotient morphisms of the form  $G \xrightarrow{p_N} G/N$  commute with the canonical morphisms between quotients, so they induce a unique morphism  $G \xrightarrow{i} \lim J$ , such that  $p_N \circ i = \pi_N$  for all  $N$  in  $\mathcal{N}$ . We will show that  $i$  is an isomorphism. Indeed, as  $G$  is compact and  $i$  is continuous,  $i$  is closed. So we just have to show bijectivity.

We have  $g \in \ker i$ , if and only if

$$p_N(g) = \pi_N(i(g)) = 1_{G/N}$$

for all  $N$  in  $\mathcal{N}$ . This is equivalent to,  $g \in N$ , for all  $N$  in  $\mathcal{N}$ . However, by Corollary 10.1.23, we have that this intersection is  $\{1_g\}$ , so  $i$  is injective.

To see that  $i$  is surjective suppose we have an element  $l$  in  $\lim J$ . We will show that there is a  $g \in G$  with  $i(g) = l$ . First suppose we have  $M$  and  $N$  in  $\mathcal{N}$  with  $M \subset N$ , let  $j$  be the canonical morphism from  $G/M$  to  $G/N$ , if we have  $f \in G$  with  $p_M(f) = \pi_M(l)$ , then we have

$$p_N(f) = jp_M(f) = j\pi_M(l) = \pi_N(l)$$

so we have  $p_M^{-1}(\pi_M(l)) \subset p_N^{-1}(\pi_N(l))$ .

Given some finite family  $(N_i)_{1 \leq i \leq n}$  of open normal subgroups of  $G$  we have that

$$M = \bigcap_{1 \leq i \leq n} N_i$$

is also an open normal subgroup of  $G$ , with  $M \subset N_i$  for  $1 \leq i \leq n$ . By the above observation this gives

$$p_M^{-1}(\pi_M(l)) \subset \bigcap_{i \leq i \leq n} p_{N_i}^{-1}(\pi_{N_i}(l)).$$

As the quotient morphism  $p_M$  is surjective we have  $p_M^{-1}(\pi_M(l)) \neq \emptyset$  so the above intersection is also non-empty.

However, the open subsets  $p_{N_i}^{-1}(\pi_{N_i}(l))$  will be closed as the group operation is continuous, so, by compactness of  $G$ , we have

$$\bigcap_{N \in \mathcal{N}} p_N^{-1}(\pi_N(l)) \neq \emptyset.$$

Now take some  $g$  in this intersection. We have  $p_N(g) = \pi_N(l)$  for all  $N \in \mathcal{N}$ , however this gives

$$\pi_N(i(g)) = p_N(g) = \pi_N(l)$$

for all  $N \in \mathcal{N}$ , that is  $i(g) = l$ , so  $i$  is surjective. That is,  $G$  is isomorphic to  $\lim J$  which is a limit of finite discrete groups.  $\square$

As above, we can generalise Proposition 10.1.24 to the pro- $p$  case and the abelian profinite case.

**Corollary 10.1.25.** *A group is pro- $p$  if and only if it is compact, Hausdorff, totally disconnected and open normal subgroups have index  $p^n$  for some  $n \in \mathbb{N}$ .*

**Corollary 10.1.26.** *A group is profinite abelian if and only if it is abelian, compact, Hausdorff and totally disconnected.*

The next lemma is necessary in the proof that a group is profinite if and only if it is algebraically compact.

**Definition 10.1.27.** *Given a group  $G$ , we define the **limit topology** on  $TG$  to be the coarsest topology on  $TG$  such that, assuming the finite groups have the discrete topology, the projection maps from  $TG$  are continuous.*

**Lemma 10.1.28.** *Suppose that we have a  $T$ -algebra  $TG \xrightarrow{h} G$ , if we equip  $G$  with the  $h$ -topology and  $TG$  with the limit topology then  $h$  is continuous.*

*Proof.* Suppose we have a basic open set  $S$  in  $G$ . Consider the set  $h^{-1}(S)$  in  $TG$ . As  $S$  is basic open, by Corollary 8.2.7 we have

$$h^{-1}(S) = \bar{S} = \hat{S}.$$

The set  $S$  is affine cofinite so there is a morphism  $G \xrightarrow{f} F$  for  $F$  finite and an element  $b \in F$  with  $S = f^{-1}(b)$ . However, letting  $TG \xrightarrow{\pi_f} G$  be the projection map corresponding to  $f$ , we have

$$\hat{S} = \pi_f^{-1}(b)$$

and as  $TG$  has the limit topology this is a basic open set in  $TG$ . So  $h$  is continuous with respect to the above topologies.  $\square$

There is only one profinite topology on a finite group, the discrete topology. Similarly, for a finite algebra  $B$ , there is only one  $T$ -algebra, the morphism

$$TB \xrightarrow{\pi_{1_B}} B$$

and the  $\pi_{1_B}$ -topology is the discrete topology on  $B$ . In the following lemma we see that this is in fact an example of a more general result.

**Lemma 10.1.29.** *A group is algebraically compact if and only if it is compact, Hausdorff and totally disconnected.*

*Proof.* First suppose we have an algebraically compact group  $G$ . By Theorem 9.2.1, the topology on  $G$  is a  $h$ -topology for some  $T$ -algebra  $TG \xrightarrow{h} G$ . By Lemma 8.2.16 the  $h$ -topology is Hausdorff. Also, by Proposition 8.2.5, the  $h$ -topology has a basis of clopen sets, so given a points  $x$  and  $y$  in  $G$ , by Hausdorffness, there is a clopen set  $U$  containing  $x$  but not  $y$ . However, we then have that  $U$  and  $U^c$  are open, with  $x \in U$ ,  $y \in U^c$ ,  $U \cap U^c = \emptyset$  and  $U \cup U^c = G$ , so  $G$  is totally separated.

Now suppose we have a subset  $S$  of  $G$  containing at least two points. By the above we can partition it into a pair of disjoint sets that are clopen in the subset topology. So  $S$  is not connected, so  $G$  is totally disconnected.

To show that it is compact suppose that, for some indexing set  $\lambda$ , we have a family,  $(S_i)_{i \in \lambda}$ , of closed subsets of  $G$  with the finite intersection property. The morphism  $h$  is surjective as it is a  $T$ -algebra, so the family  $(h^{-1}(S_i))_{i \in \lambda}$  also has the finite intersection property in  $TG$ . However, by Lemma 10.1.28,  $h$  is continuous so the family  $(h^{-1}(S_i))_{i \in \lambda}$  is closed in the limit topology on  $TG$ . With this topology  $TG$  is a profinite group, so it is compact, so we have

$$\bigcap_{i \in \lambda} (h^{-1}(S_i))_{i \in \lambda} \neq \emptyset.$$

Suppose we have a point  $x$  in the above intersection, then we will have  $h(x) \in S_i$  for all  $i \in \lambda$ , so we have

$$\bigcap_{i \in \lambda} S_i \neq \emptyset.$$

So the  $h$ -topology is compact, which gives that  $G$  is compact, Hausdorff and totally disconnected.

In the other direction suppose that the group  $G$  is compact, Hausdorff and totally disconnected. As any family of closed sets with the finite intersection property will have non-empty intersection so any family of closed affine cofinite sets with the finite intersection property will have non-empty intersection.

By Lemma 10.1.23,  $1_G$  has a neighbourhood basis of open normal subgroups. Suppose  $N$  is such an open normal subgroup of  $G$ ,  $N$  along with its cosets will form an open cover of  $G$ . By compactness this cover must have a finite subcover. However the cosets are a disjoint partition so there must in fact only be finitely many of them. So  $N$  is of finite index. This gives that  $1_G$  has a neighbourhood basis of open normal subgroups of finite index. These subgroups and their cosets are an affine cofinite basis for the topology of  $G$ . So  $G$  is algebraically compact.  $\square$

We can now apply our main result, Theorem 9.2.1, to the profinite case.

**Theorem 10.1.30.** *The category of  $T$ -algebras is isomorphic to the category of profinite groups.*

*Proof.* By Theorem 9.2.1, the category of  $T$ -algebras is isomorphic to the category of algebraically compact groups, which by combining Lemma 10.1.29 and Proposition 10.1.24 is isomorphic to the category of profinite groups.  $\square$

We also have the following corollary.

**Corollary 10.1.31.** *The algebras for the profinite completion monad are the profinite groups.*

*Proof.* This follows by combining Proposition 10.1.8 and Lemma 10.1.30.  $\square$

We can adapt Lemma 10.1.29 to the pro- $p$  case. In this case we completely avoid the problem of having to prove that a compact, Hausdorff totally disconnected group has an affine cofinite basis.

**Corollary 10.1.32.** *The category of  $T^p$ -algebras is isomorphic to the category of pro- $p$  groups.*

*Proof.* We can generalise Lemma 10.1.29 to the pro- $p$  case. So a group is algebraically compact if and only if it is compact, Hausdorff, totally separated and the open normal subgroups have order  $p^n$  for  $n \in \mathbb{N}$ . Then, by Corollary 10.1.25, we have that the algebraically compact groups are exactly the pro- $p$  groups.  $\square$

**Corollary 10.1.33.** *The algebras for the pro- $p$  completion monad are the pro- $p$  groups.*

We also adapt the above result to the abelian case. However first we must prove the following lemma.

**Corollary 10.1.34.** *For an abelian group  $G$ ,  $TG \cong T^{Ab}G$ .*

*Proof.* Let  $G$  be an abelian group. By Proposition 10.1.8 we have  $TG \cong PG$ . By the same argument as Lemma 10.1.7 we have that  $T^{Ab}G$  will be the limit of the groups  $G/N$  where  $G/N$  is finite abelian. However  $G$  is abelian so every quotient of  $G$  will be abelian, so we have  $T^{Ab}G \cong PG$ . This gives  $TG \cong T^{Ab}G$ .  $\square$

**Corollary 10.1.35.** *The  $T^{Ab}$ -algebras are the profinite abelian groups.*

*Proof.* Let  $G$  be a profinite abelian group. By Theorem 10.1.30 we have a  $T$ -algebra  $TG \xrightarrow{h} G$  such that the profinite topology on  $G$  is the  $h$ -topology. However by Corollary 10.1.34 we have that  $TG = T^{Ab}G$ . We also have that  $\eta_G$  is the unique morphism from  $G$  to  $TG$  such that

$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & TG \\ & \searrow f & \downarrow \pi_f \\ & & G \end{array}$$

commutes for all  $G \xrightarrow{f} F$  with  $F$  finite. However this gives that for all  $G \xrightarrow{g} H$  with  $H$  finite abelian the diagram

$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & T^{Ab}G \\ & \searrow g & \downarrow \pi_g \\ & & H \end{array}$$

commutes. However  $\eta_G^{Ab}$  is the unique morphism with this property, so  $\eta_G = \eta_G^{Ab}$ .

Similarly we have that  $\mu_G$  is the unique morphism such that

$$\begin{array}{ccc} TTG & & \\ \mu_A \downarrow & \searrow \Pi_f & \\ TG & \xrightarrow{\pi_f} & F \end{array}$$

commutes for all morphisms  $G \xrightarrow{f} F$  with  $F$  finite. So we have that

$$\begin{array}{ccc} T^{Ab}T^{Ab}G & & \\ \mu_G \downarrow & \searrow \Pi_g & \\ T^{Ab}G & \xrightarrow{\pi_g} & H \end{array}$$

commutes for all morphisms  $G \xrightarrow{g} H$  with  $H$  abelian. However,  $\mu_G^{Ab}$  is the unique morphism with the property, so  $\mu_G^{Ab} = \mu_G$ . This gives that  $h$  is an algebra for  $T^{Ab}$ .

In the other direction suppose we have a  $T^{Ab}$ -algebra  $T^{Ab}G \xrightarrow{h} G$ . However  $T^{Ab}G$  is a limit of abelian groups so is itself an abelian group, however the  $T^{Ab}$ -algebra  $h$  is a surjection so  $G$  is itself abelian. So, by Theorem 10.1.30, the  $h$ -topology makes  $G$  a profinite abelian group.  $\square$

## 10.2 Rings

We now adapt the above arguments to the theory of rings. Profinite rings are the obvious generalisation of profinite groups.

**Definition 10.2.1.** *A **profinite ring** is a topological ring that is a limit of finite discrete rings.*

As with groups, we will define the profinite completion of a ring as the limit of its finite quotients. To do this we make the following definitions.

**Definition 10.2.2.** Given a ring  $R$ , we let  $\mathcal{I}_R$  be the category with objects the ideals  $I$  of  $R$  such that  $R/I$  is finite and morphisms the inclusions of such ideals.

**Definition 10.2.3.** We let  $J_R$  be the functor from  $\mathcal{I}_R$  to the category **Ring** that sends an object  $I$  of  $\mathcal{I}_R$  to the ring  $R/I$ , and sends a morphism  $I \subseteq K$  in  $\mathcal{I}_R$  to the canonical morphism from  $R/I$  to  $R/K$ .

**Definition 10.2.4.** Given a ring  $R$ , the **profinite completion**  $PR$  of  $R$  is the limit in **Ring** of the functor  $J_R$ .

Consider the subcategory of **Ring** consisting of the finite rings. This is a small category closed under taking products, images and subgroups. The finite rings also satisfy the descending chain condition. So we can take this subcategory to be a subcategory of finite algebras as in Definition 2.3.3. We let  $T$  be the endofunctor of the codensity monad induced by the inclusion of finite rings in **Ring**.

**Proposition 10.2.5.** The functor  $T$  is naturally isomorphic to the functor  $P$ .

*Proof.* This is a direct generalisation of Lemma 10.1.7 and Proposition 10.1.8. □

As before we will show that the algebraically compact rings are exactly the profinite rings. First we observe that profinite rings have an interesting property that distinguishes them from profinite groups. Using an argument due to Trimble, [21], we will now see that a compact Hausdorff ring is always totally disconnected. So, a ring is profinite if and only if it is compact and Hausdorff, the condition that it is totally disconnected is redundant.

First we define the compact open topology.

**Definition 10.2.6.** Given topological spaces  $A$  and  $B$ , and subsets  $U \subset A$  and  $V \subset B$ , the set  $\text{Hom}_{U,V}(A, B)$  is the set of those continuous functions  $f \in \text{Hom}(A, B)$  such that  $f(U) \subset V$ .

**Definition 10.2.7.** Given topological spaces  $A$  and  $B$ , the **Compact Open** topology is the coarsest topology on  $\text{Hom}(A, B)$  such that for every compact subset  $U$  of  $A$  and open subset  $V$  of  $B$  the set  $\text{Hom}_{U,V}(A, B)$  is open.

**Lemma 10.2.8.** A compact Hausdorff ring is totally disconnected.

*Proof.* Let  $R$  be a compact Hausdorff ring. Consider the ring  $\text{Hom}(R, R)$ , as  $R$  is compact Hausdorff it is exponentiable, so  $\text{Hom}(R, R)$  is itself a topological ring.

Consider now the topological additive group of  $R$ . We let  $\hat{R}$  be the Pontryagin dual of the additive group of  $R$ , that is,  $\hat{R} = \text{Hom}(R, T)$ , where  $T$  is the circle group. As the additive group of  $R$  is compact and Hausdorff it is exponentiable, so  $\hat{R}$  is a topological group, and in fact, it is a topological ring. Indeed, as  $R$  is compact,  $\hat{R}$  is discrete.

The multiplication morphism  $R \times R \rightarrow R$  is continuous. So we have a continuous morphism  $R \xrightarrow{i} \text{Hom}(R, R)$ . This morphism sends an element  $r \in R$  to the morphism  $R \xrightarrow{r \cdot -} R$  so it is injective.

As  $R$  is Hausdorff  $\text{Hom}(R, R)$  is also Hausdorff. However  $R$  is compact so  $i$  maps  $R$  homeomorphically onto its image in  $\text{Hom}(R, R)$ .

The composition morphism  $\text{Hom}(R, R) \times \hat{R} \rightarrow \hat{R}$  is continuous, so its transpose  $\text{Hom}(R, R) \rightarrow \text{Hom}(\hat{R}, \hat{R})$  is also continuous. So we have continuous homomorphisms

$$\text{Hom}(R, R) \rightarrow \text{Hom}(\hat{R}, \hat{R})$$

and

$$\text{Hom}(\hat{R}, \hat{R}) \rightarrow \text{Hom}(\hat{\hat{R}}, \hat{\hat{R}}).$$

As  $\hat{\hat{R}} \cong R$  and the above morphisms are mutually inverse, we have  $\text{Hom}(R, R) \cong \text{Hom}(\hat{R}, \hat{R})$ .

By definition,  $\text{Hom}(\hat{R}, \hat{R})$  is a subspace of  $\prod_{r \in \hat{R}} \hat{R}$ . However  $\hat{R}$  is discrete so the product  $\prod_{r \in \hat{R}} \hat{R}$  is totally disconnected. Total disconnectedness is inherited by subspaces, and we have

$$R \subset \text{Hom}(R, R) \cong \text{Hom}(\hat{R}, \hat{R}) \subset \prod_{r \in \hat{R}} \hat{R}$$

so  $R$  is totally disconnected. □

We must also extend Lemma 10.1.22 to the **Ring** case.

**Lemma 10.2.9.** *If  $R$  is a compact Hausdorff totally disconnected ring, then for some indexing set  $\lambda$ ,  $R$  has a basis of open ideals  $(I_i)_{i \in \lambda}$  with*

$$\bigcap_{i \in \lambda} I_i = \{0_R\}.$$

*Proof.* Let  $R$  be a compact Hausdorff ring. By Lemma 10.1.22 the additive group of  $R$  has a basis of open subgroups. We will use this to construct a basis of open ideals. Given one such open subgroup  $U$  of  $R$  consider the set

$$I = \{x \in R \mid R \cdot x \cdot R \subset U\}.$$

This  $I$  is an ideal and as we have  $1_R \in R$  then  $I \subset U$ . Consider a point  $x \in I$ , we will construct an open set  $V$  with  $x \in V \subset I$ . For each pair  $p, q \in R$  we have  $p \cdot x \cdot q \in U$ . We have a continuous multiplication morphism  $R \times I \times R \rightarrow R$  where  $I$  has the subspace topology. So, as  $U$  is open, by continuity there are open sets,  $U_{p,q}$ ,  $W_{p,q}$  of  $R$  and  $V_{p,q}$  of  $I$  with  $p \in U_{p,q}$ ,  $x \in V_{p,q}$  and  $q \in W_{p,q}$  such that

$$U_{p,q} \cdot V_{p,q} \cdot W_{p,q} \subset U.$$

Now the  $U_{p,q} \times W_{p,q}$  cover  $R \times R$ , however as it is compact, so, for some finite index  $I$ , there are points  $(p_i, q_i) \in R \times R$ , for  $i \in I$ , such that the sets  $U_{p_i, q_i} \times W_{p_i, q_i}$  cover  $R \times R$ . Consider the intersection

$$V = \bigcap_{1 \leq i \leq n} V_{p_i, q_i}.$$

This will be a finite intersection of open sets so will itself be open and we have  $x \in V \subset I$ , so  $I$  is itself open. As we have  $I \subset U$ , then the intersection of these ideals is a subset of the intersection of the additive subgroups. However by Lemma 10.1.22, this intersection is trivial, so the intersection of the ideals is equal to  $\{0_R\}$ . □

**Proposition 10.2.10.** *A ring is profinite if and only if it is compact and Hausdorff.*



*Proof.* This follows as a direct generalisation of Proposition 10.1.24 using Lemma 10.2.8.

□

**Proposition 10.2.11.** *A ring is algebraically compact if and only if it is compact Hausdorff.*

*Proof.* This follows as a direct generalisation of Lemma 10.1.29.

□

Finally we apply Theorem 9.2.1 to the ring case.

**Theorem 10.2.12.** *The category of  $T$ -algebras is isomorphic to the category of profinite rings.*

*Proof.* By Theorem 9.2.1, the category of  $T$ -algebras is isomorphic to the category of algebraically compact rings, which by combining Propositions 10.2.11 and 10.2.10 is isomorphic to the category of profinite rings.

□

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